Learning Objectives

A student will be able to:

- Learn how to apply definite integrals to several applications from physics, engineering, and applied mathematics such as work, fluids statics, and probability.

In this section we will show how the definite integral can be used in different applications. Some of the concepts may sound new to the reader, but we will explain what you need to comprehend as we go along. We will take three applications: The concepts of work from physics, fluid statics from engineering, and the normal probability from statistics.

Work

Work in physics is defined as the product of the force and displacement. Force and displacement are vector quantities, which means they have a direction and a magnitude. For example, we say the compressor exerts a force of $200 \text{Newtons} \ (\text{N})$ upward. The magnitude here is 200N and the direction is upward. Lowering a book from an upper shelf to a lower one by a distance of 0.5meters away from its initial position is another example of the vector nature of the displacement. Here, the magnitude is 0.5m and the direction is downward, usually indicated by a minus sign, i.e., a displacement of -0.5m. The product of those two vector quantities (called the ***inner product***, see Chapter 10) gives the work done by the force. Mathematically, we say

$$W=Fd,$$

where F is the force and d is the displacement. If the force is measured in Newtons and distance is in meters, then work is measured in the units of energy which is in joules (J).

Example 1:

You push an empty grocery cart with a force of 44N for a distance of 12meters. How much work is done by you (the force)?

Solution:

Using the formula above,

$$W=Fd=(44)(12)=528 \text{ J}.$$

Example 2:

A librarian displaces a book from an upper shelf to a lower one. If the vertical distance between the two shelves is 0.5meters and the weight of the book is 5Newtons . How much work is done by the librarian?

Solution:

In order to be able to lift the book and move it to its new position, the librarian must exert a force that is at least equal to the weight of the book. In addition, since the displacement is a vector quantity, then the direction must be taken into account. So,

$d=-0.5$ meters.
Thus

$W=Fd=(5)(-0.5)=-2.5$ J.
Here we say that the work is negative since there is a loss of gravitational potential energy rather than a gain in energy. If the book is lifted to a higher shelf, then the work is positive, since there will be a gain in the gravitational potential energy.

Example 3:

A bucket has an empty weight of 23N. It is filled with sand of weight 80N and attached to a rope of weight 5.1N/m. Then it is lifted from the floor at a constant rate to a height 32meters above the floor. While in flight, the bucket leaks sand grains at a constant rate, and by the time it reaches the top no sand is left in the bucket. Find the work done:

1. by lifting the empty bucket;
2. by lifting the sand alone;
3. by lifting the rope alone;
4. by the lifting the bucket, the sand, and the rope together.

Solution:

1. *The empty bucket.* Since the bucket's weight is constant, the worker must exert a force that is equal to the weight of the empty bucket. Thus

$W=Fd=(23)(+32)=736$ J.
2. *The sand alone.* The weight of the sand is decreasing at a constant rate from 80N to 0N over the $32-$meter lift. When the bucket is at $x$meters above the floor, the sand weighs
$F(x)=[\text{original weight of sand}][\text{proportion left at elevation } x]=80(32-x32)=80(1-x32)=80-2.5x$ N.
The graph of $F(x)=80-2.5x$ represents the variation of the force with height x (Figure 23). The work done corresponds to computing the area under the force graph.

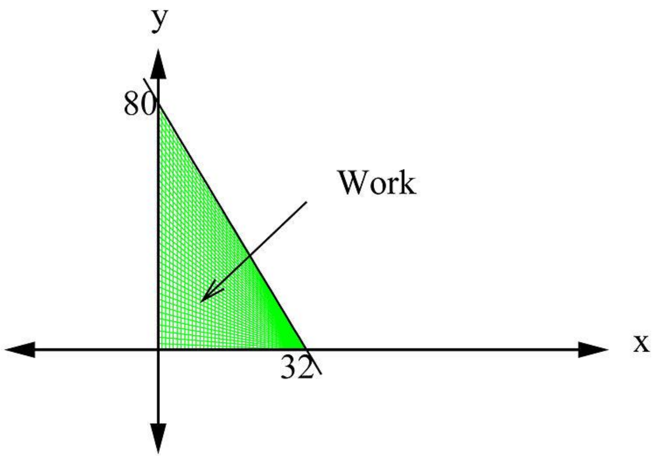


Figure 23

Thus the work done is

$$W=\int_a^b F(x)dx=\int_0^{32}[80-2.5x]dx=[80x-\frac{2.5}{2}x^2]_0^{32}=1280 \text{ J}.$$

3. *The rope alone.* Since the weight of the rope is $5.1 N/m$ and the height is $32 meters$, the total weight of the rope from the floor to a height of $32 meters$ is

$$(5.1)(32)=163.2 \text{ N}.$$

But since the worker is constantly pulling the rope, the rope's length is decreasing at a constant rate and thus its weight is also decreasing as the bucket being lifted. So at $x meters$, the $(32-x) meters$ there remain to be lifted of weight $F(x)=(5.1)(32-x) N$. Thus the work done to lift the weight of the rope is

$$W_W=\int_0^{32} F(x)dx=\int_0^{35}(5.1)(32-x)dx=(5.1)[32x-\frac{x^2}{2}]_0^{32}=2611.2 \text{ J}.$$

4. *The bucket, the sand, and the rope together*. Here we are asked to sum all the work done on the empty bucket, the sand, and the rope. Thus

$$W_{total}=736+1280+2611.2=4627.2 \text{ J}.$$

Fluid Statics: Pressure

You have probably studied that ***pressure*** is defined as the force per area

$$P=\frac{F}{A},$$

which has the units of Pascals (Pa) or Newtons per meter squared, $Pa=N/m^2$. In the study of fluids, such as water pressure on a dam or water pressure in the ocean at a depth h, another equivalent formula can be used. It is called the ***liquid pressure*** P at depth h:

$$P=wh.$$

where w is the *weight density*, which is the weight of the column of water per unit volume. For example, if you are diving in a pool, the pressure of the water on your body can be measured by calculating the total weight that the column of water is exerting on you times your depth. Another way to express this formula, the weight density w, is defined as

$w=\rho g$,

where ρ is the density of the fluid and g is the acceleration due to gravity (which is $g=9.8\ m/sec^2$ on Earth). The pressure then can be written as

$P=wh=\rho gh$.

Example 4:

What is the total pressure experienced by a diver in a swimming pool at a depth of 2meters ?

Solution

First we calculate the fluid pressure the water exerts on the diver at a depth of 2meters :

$P=\rho gh$.

The density of water is $\rho=1000 kg/m^3$, thus

$P=(1000)(9.8)(2)=19600\ Pa$.

The total pressure on the diver is the pressure due to the water plus the atmospheric pressure. If we assume that the diver is located at sea-level, then the atmospheric pressure at sea level is about $10^5 Pa$. Thus the total pressure on the diver is

$P_{total}=P_{water}+P_{atm}=19600+10^5=119600=1.196\times10^5\ Pa$.

Example 5:

What is the fluid pressure (excluding the air pressure) and force on the top of a flat circular plate of radius 3meters that is submerged horizontally at a depth of 10meters ?

Solution:

The density of water is $\rho=1000 kg/m^3$. Then

$P=\rho gh=(1000)(9.8)(10)=98000\ Pa$.

Since the force is $F=PA$, then

$F=PA=P\cdot\pi r^2=(98000)(\pi)(3)^2=2.77\times10^6\ N$.

As you can see, it is easy to calculate the fluid force on a horizontal surface because each point on the surface is at the same depth. The problem becomes a little complicated when we want to calculate the fluid force or pressure if the surface is vertical. In this situation, the pressure is not constant at every point because the depth is not constant at each point. To find the fluid force or pressure on a vertical surface we must use calculus.

The Fluid Force on a Vertical Surface

Suppose a flat surface is submerged vertically in a fluid of weight density w and the submerged portion of the surface extends from $x=a$ to $x=b$ along the vertical $x-$axis, whose positive direction is taken as downward. If $L(x)$ is the width of the surface and $h(x)$ is the depth of point x, then the ***fluid force*** F is defined as
$F=\int_a^b wh(x)L(x)dx$.

Example 6:

A perfect example of a vertical surface is the face of a dam. We can picture it as a rectangle of a certain height and certain width. Let the height of the dam be 100meters and of width of 300meters. Find the total fluid force exerted on the face if the top of the dam is level with the water surface (Figure 24).

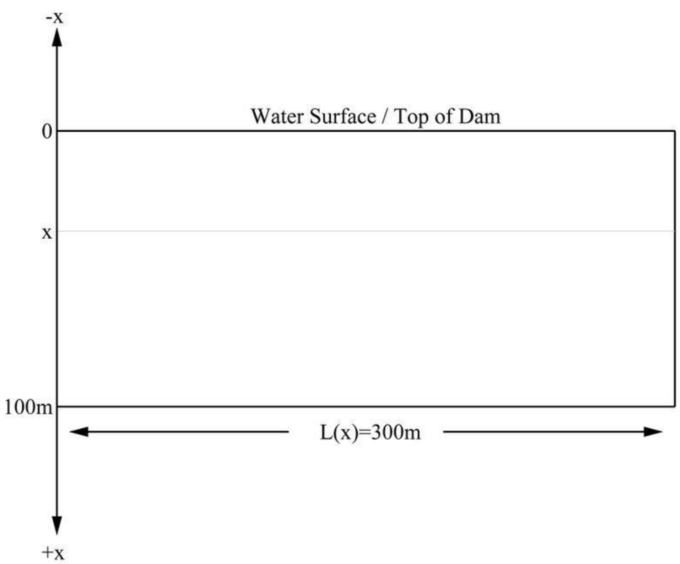


Figure 24

Solution:

Let $x=$ the depth of the water. At an arbitrary point x on the dam, the width of the dam is $L(x)=300m$ and the depth is $h(x)=xm$. The weight density of water is $w_{water}=\rho g=(1000)(9.8)=9800\ N/m^2$.
Using the fluid force formula above,

$$F=\int_a^b wh(x)L(x)dx=\int_0^{100}(9800)(x)(300)dx=2.94\times10^6\int_0^{100}x\,dx=2.94\times10^6\left[\frac{x^2}{2}\right]_0^{100}=1.47\times10^{10}\ N.$$

Normal Probabilities

If you were told by the postal service that you will receive the package that you have been waiting for sometime tomorrow, what is the probability that you will receive it sometime between 3:00 PM and 5:00 PM if you know that the postal service's hours of operations are between 7:00 AM to 6:00 PM?

If the hours of operations are between 7 AM to 6 PM, this means they operate for a total of 11hours. The interval between 3 PM and 5 PM is 2hours, and thus the probability that your package will arrive is
$P=\frac{2\text{ hours}}{11\text{ hours}}=0.182=18.2\%$
So there is a probability of 18.2% that the postal service will deliver your package sometime between the hours of 3 PM and 5 PM (or during any $2-\text{hour}$ interval). That is easy enough. However, mathematically, the situation is not that simple.
The $11-\text{hour}$ interval and the $2-\text{hour}$interval contain an infinite number of times. So how can one infinity over another infinity produce a probability of 18.2%? To resolve this issue, we represent the total probability of the $11-\text{hour}$ interval as a rectangle of area 1 (Figure 25).

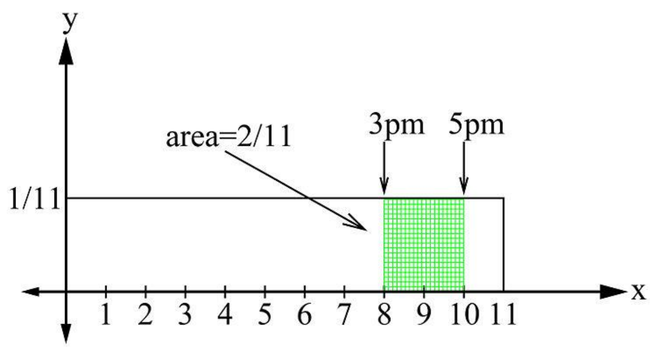


Figure 25

Looking at the $2-\text{hour}$ interval, we can see that it is equal to $\frac{2}{11}$ of the total rectangular area 1. This is why it is convenient to represent probabilities as areas. But since areas can be defined by definite integrals, we can also define the probability associated with an interval $[a,b]$ by the definite integral
$P=\int_a^b f(x)dx$,
where $f(x)$ is called the ***probability density function*** (pdf). One of the most useful probability density functions is the ***normal curve*** or the ***Gaussian curve*** (and sometimes the ***bell curve***) (Figure 26). This function enables us to describe an entire population based on statistical measurements taken from a small sample of the population. The only measurements needed are the mean (μ) and the standard deviation (σ). Once those two numbers are known, we can easily find the normal curve by using the following formula.

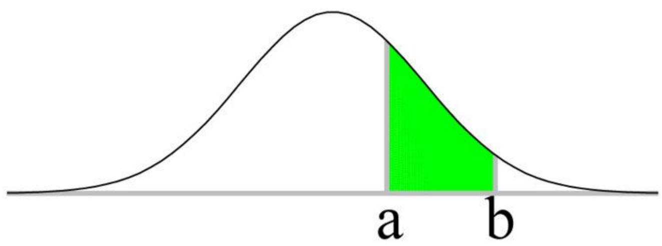


Figure 26

The Normal Probability Density Function

The Gaussian curve for a population with mean μ and standard deviation σ is given by
$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$,
where the factor $1/(\sigma\sqrt{2\pi})$ is called the *normalization constant.* It is needed to make the probability over the entire space equal to 1. That is,
$P(-\infty<x<\infty)=\int_{-\infty}^{+\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}=1$.

Example 7:

Suppose that boxes containing 100 tea bags have a mean weight of $10.2\,\text{ounces}$ each and a standard deviation of $0.1\,\text{ounce}$.

1. What percentage of all the boxes is expected to weigh between 10 and $10.5\,\text{ounces}$?
2. What is the probability that a box weighs less than $10\,\text{ounces}$?
3. What is the probability that a box will weigh exactly $10\,\text{ounces}$?

Solution:

1. Using the normal probability density function,

$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$.
Substituting for $\mu=10.2$ and $\sigma=0.1$, we get
$f(x)=\frac{1}{(0.1)\sqrt{2\pi}}e^{-(x-10.2)^2/(2(0.1)^2)}$.
The percentage of all the tea boxes that are expected to weight
between 10 and 10.5 ounces can be calculated as
$P(10\le x\le 10.5)=\int_{10}^{10.5}\frac{1}{(0.1)\sqrt{2\pi}}e^{-(x-10.2)^2/(2(0.1)^2)}dx$.
The integral of e^{x^2} does not have an elementary anti-derivative and therefore cannot be evaluated by standard techniques. However, we can use numerical techniques, such as The Simpson’s Rule or The Trapezoid Rule, to find an approximate (but very accurate) value. Using the programing feature of a scientific calculator or, mathematical software, we eventually get
$\int_{10}^{10.5}\frac{1}{(0.1)\sqrt{2\pi}}e^{-(x-10.2)^2/(2(0.1)^2)}dx\approx 0.976$.
That is,

$P(10\le x\le 10.5)\approx 97.6\%$.

Technology Note: To make this computation with a graphing calculator of the TI-83/84 family, do the following:

- From the **[DISTR]** menu (Figure 27) select option 2, which puts the phrase "normalcdf" in the home screen. Add lower bound, upper bound, mean, standard deviation, separated by commas, close the parentheses, and press **[ENTER]**. The result is shown in Figure 28.

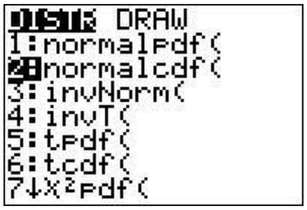


Figure 27

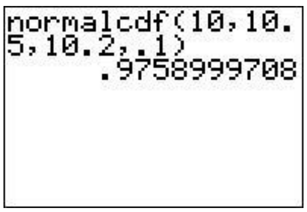


Figure 28

2. For the probability that a box weighs less than 10.2ounces, we use the area under the curve to the left of $x=10.2$. Since the value of $f(9)$is very small (less than a billionth),
$f(9)=\frac{1}{(0.1)\sqrt{2\pi}}e^{-(9-10.2)^2/(2(0.1)^2)}dx=1.35\times10^{-32}$,
getting the area between 9 and 10 will yield a fairly good answer. Integrating numerically, we get
$P(9\leq x\leq 10)P(9\leq x\leq 10.2)=\int_9^{10}\frac{1}{(0.1)\sqrt{2\pi}}e^{-(x-10.2)^2/(2(0.1)^2)}dx\approx0.02275=2.28\%$,
which says that we would expect 2.28% of the boxes to weigh less than 10ounces.
3. Theoretically the probability here will be exactly zero because we will be integrating from 10 to 10, which is zero. However, since all scales have some error (call it ϵ), practically we would find the probability that the weight falls between $10-\epsilon$ and $10+\epsilon$.
Example 8:

An Intelligence Quotient or IQ is a score derived from different standardized tests attempting to measure the level of intelligence of an adult human being. The average score of the test is 100 and the standard deviation is 15.

1. What is the percentage of the population that has a score between $85$and 115?
2. What percentage of the population has a score above 140?

Solution:

1. Using the normal probability density function,

$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$,
and substituting $\mu=100$ and $\sigma=15$,
$f(x)=\frac{1}{15\sqrt{2\pi}}e^{-(x-100)^2/(2(15)^2)}$.
The percentage of the population that has a score between 85 and $115$is
$P(85\le x\le 115)=\int_{85}^{115}\frac{1}{15\sqrt{2\pi}}e^{-(x-100)^2/(2(15)^2)}$.
Again, the integral of e^{-x^2} does not have an elementary anti-derivative and therefore cannot be evaluated. Using the programing feature of a scientific calculator or a mathematical computer software, we get
$\int_{85}^{115}\frac{1}{15\sqrt{2\pi}}e^{-(x-100)^2/(2(15)^2)}dx\approx 0.68$.
That is,

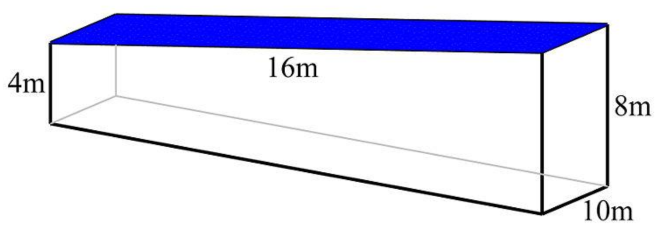
$P(85\le x\le 115)\approx 68\%$.
Which says that 68% of the population has an IQ score between 85 and 115.
2. To measure the probability that a person selected randomly will have an IQ score above 140,
$P(x\ge 140)=\int_{140}^{\infty}\frac{1}{15\sqrt{2\pi}}e^{-(x-100)^2/(2(15)^2)}dx$.
This integral is even more difficult to integrate since it is an improper integral. To avoid the messy work, we can argue that since it is extremely rare to meet someone with an IQ score of over 200, we can approximate the integral from 140 to 200. Then
$P(x\ge 140)\approx\int_{140}^{200}\frac{1}{15\sqrt{2\pi}}e^{-(x-100)^2/(2(15)^2)}dx$.
Integrating numerically, we get

$P(x\ge 140)\approx 0.0039$.
So the probability of selecting at random a person with an IQ score above 140 is 0.39%. That's about one person in every 250 individuals!

Review Questions

1. A particle moves along the $x-$axis by a force $F(x)=\frac{1}{x^2+1}$.If the particle has already moved a distance of $10\,meters$ from the origin, what is the work done by the force?
2. A force of $\cos(\frac{\pi x}{2})$ acts on an object when it is $x\,meters$ away from the origin. How much work is done by this force in moving the object from $x=1$ to $x=5\,meters$?
3. In physics, if the force on an object varies with distance then work done by the force is defined as (see Example 5.15) $W=\int_a^b F(r)dr$.That is, the work done corresponds to computing the area under the force graph. For example, the strength of the gravitational field varies with the distance r from the Earth's center. If a satellite of mass m is to be launched into space, then the force experienced by the satellite during and after launch is $F(r)=G\frac{mM}{r^2}$,where $M=6\times 10^{24}kg$ is the mass of the Earth and $G=6.67\times 10^{-11}\frac{Nm^2}{kg^2}$ is the Universal Gravitational Constant. If the mass of the satellite is $1000kg$ and we wish to lift it to an altitude of $35{,}780km$ above the Earth's surface, how much work is needed to lift it? (Radius of Earth is $6370km$.)

4. *Hook's Law* states that when a spring is stretched x units beyond its natural length it pulls back with a force $F(x)=kx$, where k is called the *spring constant* or the *stiffness* constant. To calculate the work required to stretch the spring a length x we use $W=\int_a^b F(x)dx$, where a is the initial displacement of the spring ($a=0$ if the spring is initially unstretched) and b is the final displacement. A force of $5N$ is exerted on a spring and stretches it $1m$ beyond its natural length.
   a. Find the spring constant k.
   b. How much work is required to stretch the spring $1.8m$ beyond its natural length?
5. When a force of $30N$ is applied to a spring, it stretches it from a length of $12cm$ to $15cm$. How much work will be done in stretching the spring from $12cm$ to $20cm$? (Hint: read the first part of problem #4 above.)
6. A flat surface is submerged vertically in a fluid of weight density w. If the weight density w is doubled, is the force on the plate also doubled? Explain.
7. The bottom of a rectangular swimming pool, whose bottom is an inclined plane, is shown below. Calculate the fluid force on the bottom of the pool when it is filled completely with water.



8. Suppose $f(x)$ is the probability density function for the lifetime of a manufacturer's light bulb, where x is measured in hours. Explain the meaning of each integral.
   a. $\int_{1000}^{5000} f(x)dx$
   b. $\int_{3000}^{\infty} f(x)dx$
9. The length of time a customer spends waiting until his/her entree is served at a certain restaurant is modeled by an exponential density function with an average time of 8 minutes.
   a. What is the probability that a customer is served in the first 3 minutes?
   b. What is the probability that a customer has to wait more than 10 minutes?
10. The average height of an adult female in Los Angeles is 63.4 inches (5 feet 3.4 inches) with a standard deviation of 3.2 inches.
    a. What is the probability that a female's height is less than 63.4 inches?
    b. What is the probability that a female's height is between 63 and 65 inches?

c. What is the probability that a female’s height is more than 6feet?
d. What is the probability that a female’s height is exactly 5feet?