Learning Objectives

A student will be able to:

- Learn how to find the area of a surface that is generated by revolving a curve about an axis or a line.

In this section we will deal with the problem of finding the area of a surface that is generated by revolving a curve about an axis or a line. For example, the surface of a sphere can be generated by revolving a semicircle about its diameter (Figure 19) and the circular cylinder can be generated by revolving a line segment about any axis that is parallel to it (Figure 20).

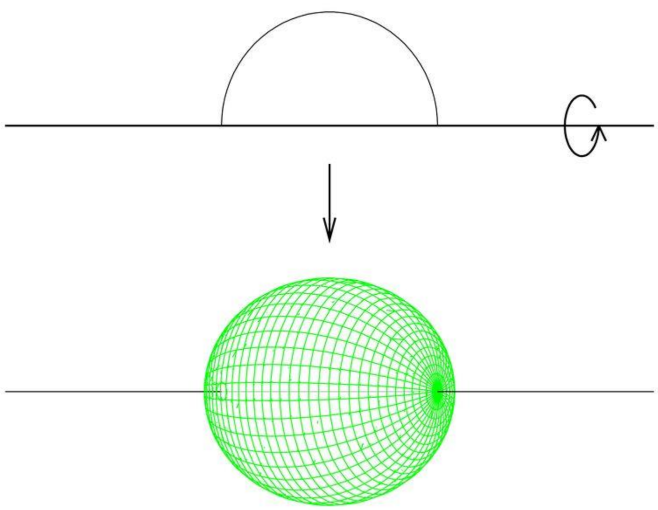

Figure 19

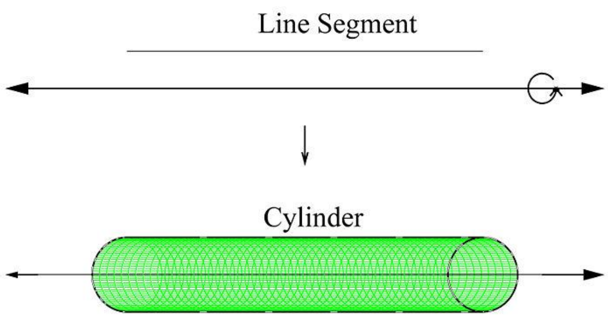

Figure 20

Area of a Surface of Revolution

If f is a smooth and non-negative function in the interval $[a,b]$, then the surface area S generated by revolving the curve $y=f(x)$ between $x=a$ and $x=b$ about the $x-$axis is defined by

$$S=\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx=\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

Equivalently, if the surface is generated by revolving the curve about the $y-$axis between $y=c$ and $y=d$, then

$$S=\int_c^d 2\pi g(y)\sqrt{1+[g'(y)]^2}\,dy=\int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy.$$

Example 1:

Find the surface area that is generated by revolving $y=x^3$ on $[0,2]$about the $x-$axis (Figure 21).

Solution:

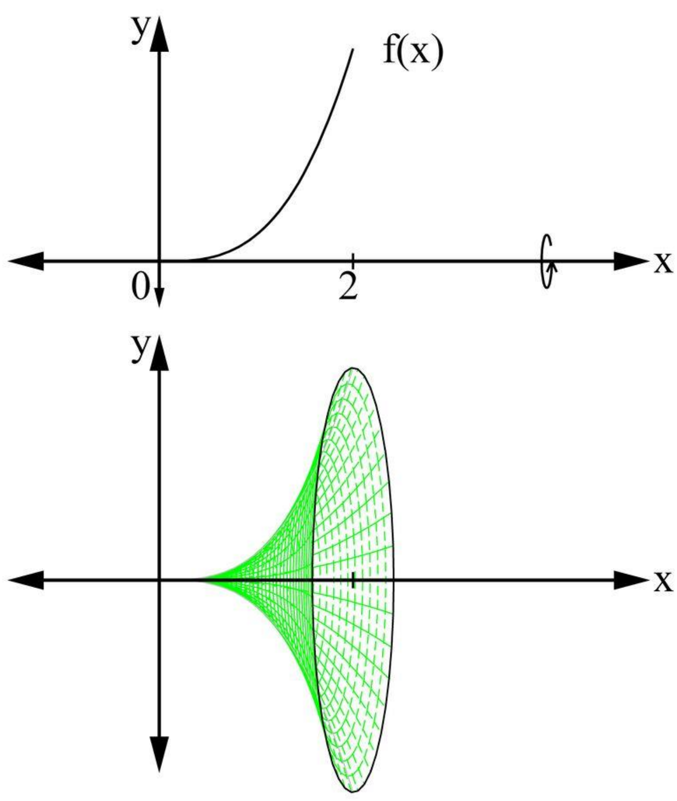

Figure 21

The surface area S is

$$S=\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^2 2\pi x^3\sqrt{1+(3x^2)^2}\,dx=2\pi\int_0^2 x^3(1+9x^4)^{1/2}\,dx.$$

Using u−substitution by letting $u=1+9x^4$,
$S=2\pi\int_1^{145} u^{1/2}\frac{du}{36}=\frac{2\pi}{36}\left[\frac{2}{3}u^{3/2}\right]_1^{145}=\frac{2\pi}{36}\cdot\frac{2}{3}\left[(145)^{3/2}-1\right]\approx\frac{4\pi}{108}[1745]\approx 203$

Example 2:

Find the area of the surface generated by revolving the graph of $f(x)=x^2$ on the interval $[0,\sqrt{3}]$ about the y−axis (Figure 22).

Solution:

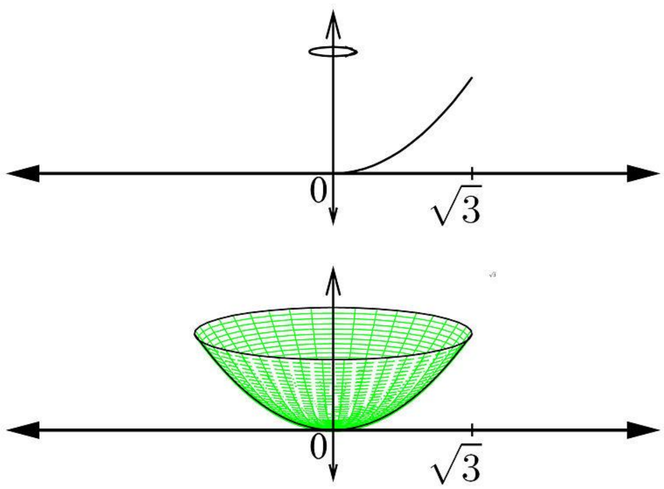

Figure 22

Since the curve is revolved about the y−axis, we apply
$S=\int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy.$
So we write $y=x^2$ as $x=\sqrt{y}$. In addition, the interval on the x−axis $[0,\sqrt{3}]$ becomes $[0,3]$. Thus
$S=\int_0^3 2\pi\sqrt{y}\sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2}\,dy.$
Simplifying,

$S=\pi\int_0^3\sqrt{4y+1}\,dy.$
With the aid of u−substitution, let $u=4y+1$,
$S=\frac{\pi}{4}\int_1^{13} u^{1/2}\,du=\frac{\pi}{6}\left[(13)^{3/2}-1\right]=\frac{\pi}{6}[46.88-1]\approx 24$

Review Questions

In problems #1 - 3 find the area of the surface generated by revolving the curve about the x−axis.

1. $y=3x, 0\le x\le 1$
2. $y=\sqrt{x}, 1\le x\le 9$
3. $y=\sqrt{4-x^2}, -1\le x\le 1$

In problems #4–6 find the area of the surface generated by revolving the curve about the y−axis.

4. $x=7y+2, 0\le y\le 3$
5. $x=y^3, 0\le y\le 8$
6. $x=\sqrt{9-y^2}, -2\le y\le 2$
7. Show that the surface area of a sphere of radius r is $4\pi r^2$.
8. Show that the lateral area S of a right circular cone of height h and base radius r is $S=\pi r\sqrt{r^2+h^2}$.