Learning Objectives

A student will be able to:

- Learn how to find the length of a plane curve for a given function.

In this section will consider the problem of finding the length of a plane curve. Formulas for finding the arcs of circles appeared in early historical records and they were known to many civilizations. However, very little was known about finding the lengths of general curves, such as the length of the curve $y=x^2$ in the interval $[0,2]$, until the discovery of calculus in the seventeenth century.

In calculus, we define an ***arc length*** as the length of a plane curve $y=f(x)$ over an interval $[a,b]$ (Figure 17). When the curve $f(x)$ has a continuous first derivative f' on $[a,b]$, we say that f is a smooth function (or smooth curve) on $[a,b]$.

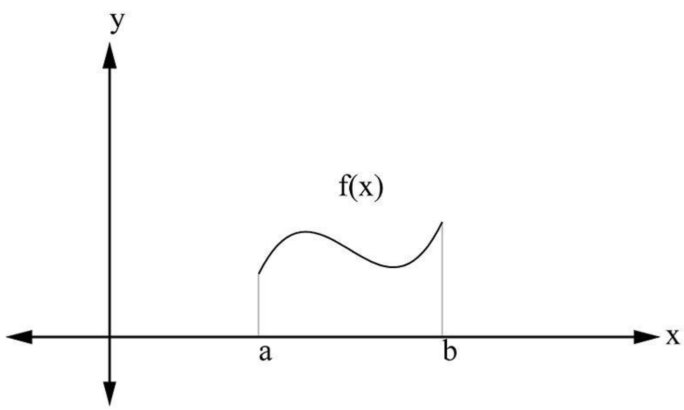


Figure 17

The Arc Length Problem

If $y=f(x)$ is a smooth curve on the interval $[a,b]$, then the arc length $L$of this curve is defined as

$$L=\int_a^b \sqrt{1+[f'(x)]^2}\,dx=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

Example 1:

Find the arc length of the curve $y=x^{\frac{3}{2}}$ on $[1,3]$ (Figure 18).

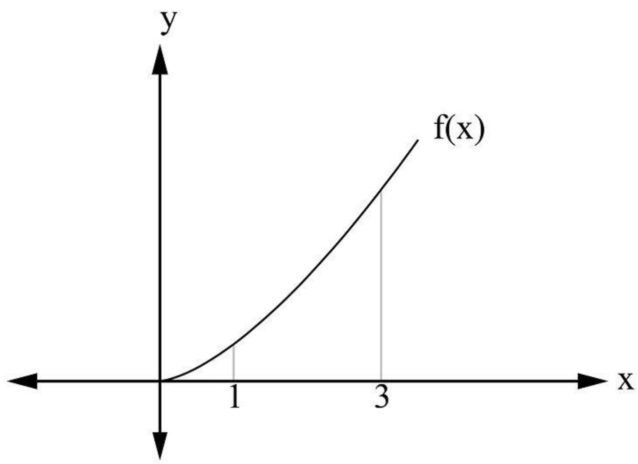


Figure 18

Solution:

Since $y=x^{3/2}$,
$\frac{dy}{dx}=\frac{3}{2}x^{1/2}$.
Using the formula above, we get

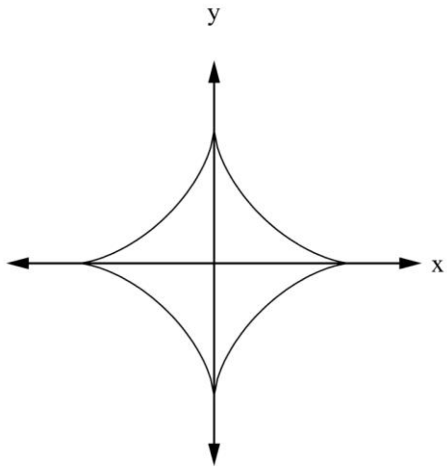
$$\int_a^b \sqrt{1+[f'(x)]^2}\,dx=\int_1^3 \sqrt{1+\left[\frac{3}{2}x^{1/2}\right]^2}\,dx=\int_1^3 \sqrt{1+\frac{9}{4}x}\,dx.$$

Using $u-$substitution by letting $u=1+\frac{9}{4}x$, then $du=\frac{9}{4}dx$. Substituting, and remembering to change the limits of integration,
$$L=\frac{4}{9}\int_{13/4}^{31/4} \sqrt{u}\,du=\frac{8}{27}\left[u^{3/2}\right]_{13/4}^{31/4}\approx 4.65.$$

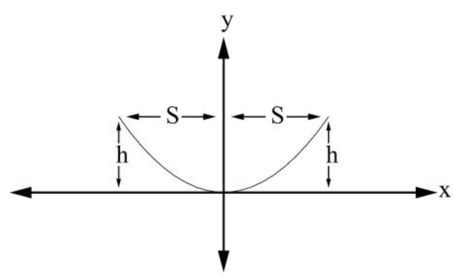
Review Questions

1. Find the arc length of the curve $y=\frac{(x^2+2)^{3/2}}{3}$ on $[0,3]$.
2. Find the arc length of the curve $x=\frac{1}{6}y^3+\frac{1}{2y}$ on $y\in[1,2]$.
3. Integrate $x=\int_0^y \sqrt{\sec^4 t-1}\,dt$, $-\frac{\pi}{4}\le y\le\frac{\pi}{4}$.

4. Find the length of the curve shown in the figure below. The shape of the graph is called the *asteroid* because it looks like a star. The equation of its graph is $x^{2/3}+y^{2/3}=1$.



5. The figure below shows a suspension bridge. The cable has the shape of a parabola with equation $kx^2=y$. The suspension bridge has a total length of $2S$ and the height of the cable is h at each end. Show that the total length of the cable



is $L=2\int_0^S \sqrt{1+\frac{4h^2}{S^4}x^2}\,dx$.