Learning Objectives

- Learn the basic concepts of volume and how to compute it with a given cross-section
- Learn how to compute volume by the *disk method*
- Learn how to compute volume by the *washer method*
- Learn how to compute volume by *cylindrical shells*

In this section, we will use definite integrals to find volumes of different solids.

The Volume Formula

A circular cylinder can be generated by translating a circular disk along a line that is perpendicular to the disk (Figure 5). In other words, the cylinder can be generated by moving the cross-sectional area A (the disk) through a distance h. The resulting volume is called the ***volume of solid*** and it is defined to be
$V=Ah$.

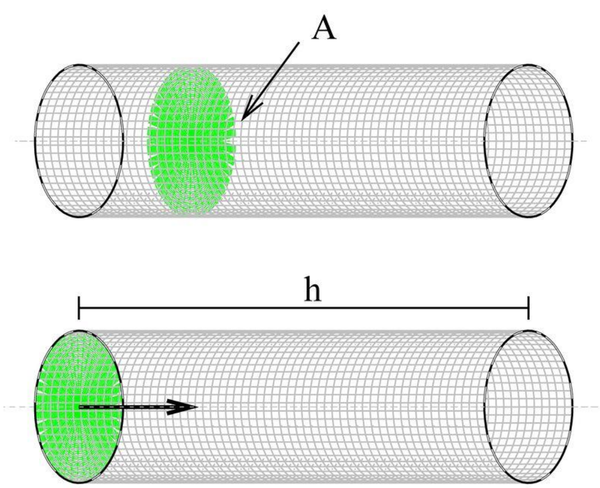

Figure 5

The volume of solid does not necessarily have to be circular. It can take any arbitrary shape. One useful way to find the volume is by a technique called "slicing." To explain the idea, suppose a solid S is positioned on the $x-$axis and extends from points $x=a$ to $x=b$ (Figure 6).

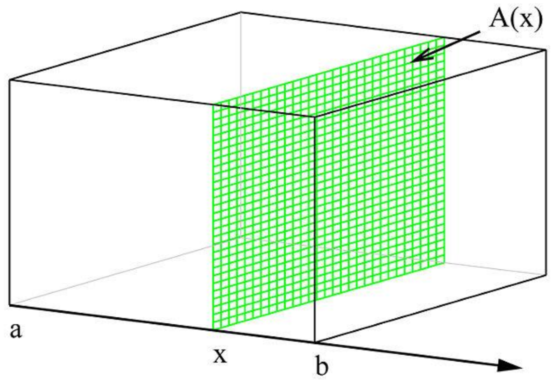

Figure 6

Let $A(x)$ be the cross-sectional area of the solid at some arbitrary point x. Just like we did in calculating the definite integral in the previous chapter, divide the interval $[a,b]$ into n sub-intervals and with widths

$\Delta x_1, \Delta x_2, \Delta x_3, \ldots, \Delta x_n$.

Eventually, we get planes that cut the solid into n slices

$S_1, S_2, S_3, \ldots, S_n$.

Take one slice, S_k. We can approximate slice S_k to be a rectangular solid with thickness Δx_k and cross-sectional area $A(x_k)$. Thus the volume V_k of the slice is approximately

$V_k \approx A(x_k)\Delta x_k$.

Therefore the volume V of the entire solid is approximately

$V = V_1 + V_2 + \ldots + V_n \approx \sum_{k=1}^{n} A(x_k)\Delta x_k$.

If we use the same argument to derive a formula to calculate the area under the curve, let us increase the number of slices in such a way that $\Delta x_k \to 0$. In this case, the slices become thinner and thinner and, as a result, our approximation will get better and better. That is,

$V = \lim_{\Delta x \to 0} = \sum_{k=1}^{n} A(x_k)\Delta x_k$.

Notice that the right-hand side is just the definition of the definite integral. Thus

$V = \lim_{\Delta x \to 0} = \sum_{k=1}^{n} A(x_k)\Delta x_k = \int_a^b A(x)\,dx$.

The Volume Formula (*Cross-section perpendicular to the* $x-$axis*)*

Let S be a solid bounded by two parallel planes perpendicular to the $x-$axis at $x=a$ and $x=b$. If each of the cross-sectional areas in $[a,b]$are perpendicular to the $x-$ axis, then the volume of the solid is given by

$V = \int_a^b A(x)\,dx$.

where $A(x)$ is the area of a cross section at the value of x on the $x-$axis.

The Volume Formula (*Cross-section perpendicular to the* $y-$*axis*)

Let S be a solid bounded by two parallel planes perpendicular to the $y-$axis at $y=c$ and $y=d$. If each of the cross-sectional areas in $[c,d]$are perpendicular to the $y-$axis, then the volume of the solid is given by
$V=\int_c^d A(y)dy.$
where $A(y)$ is the area of a cross section at the value of y on the $y-$axis.

Example 1:

Derive a formula for the volume of a pyramid whose base is a square of sides a and whose height (altitude) is h.

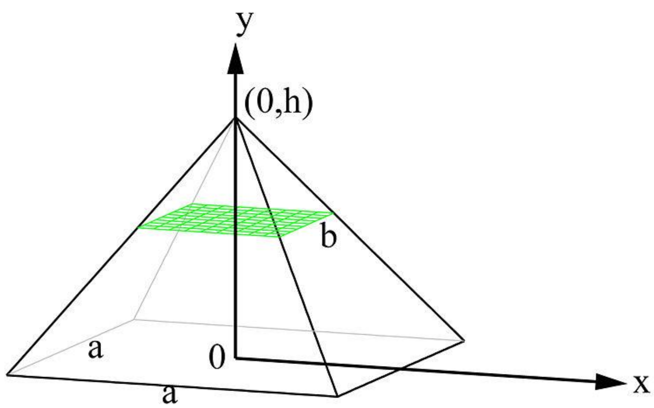

Figure 7a

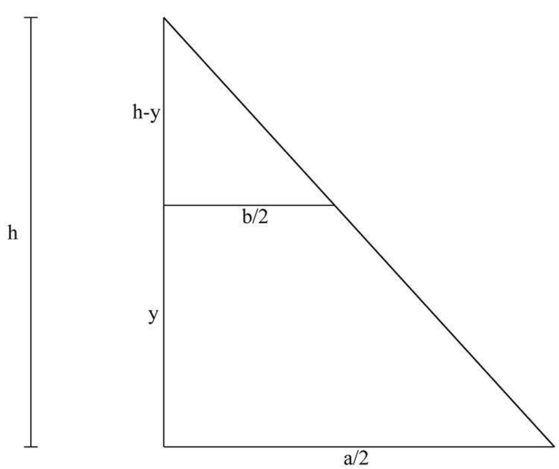

Figure 7b

Solution:

Let the $y-$axis pass through the apex of the pyramid, as shown in Figure (7a). At any point y in the interval $[0,h]$, the cross-sectional area is a square. If b is the length of the sides of any arbitrary square, then, by similar triangles (Figure 7b),

$$\frac{\frac{1}{2}b}{\frac{1}{2}a} = \frac{h-y}{h}, \quad b = \frac{a}{h}(h-y).$$

Since the cross-sectional area at y is $A(y)=b^2$,

$$A(y)=b^2=\frac{a^2}{h^2}(h-y)^2.$$

Using the volume formula,

$$V=\int_c^d A(y)\,dy=\int_0^h \frac{a^2}{h^2}(h-y)^2\,dy=\frac{a^2}{h^2}\int_0^h (h-y)^2\,dy.$$

Using $u-$substitution to integrate, we eventually get

$$V=\frac{a^2}{h^2}\left[-\frac{1}{3}(h-y)^3\right]_0^h=\frac{1}{3}a^2h.$$

Therefore the volume of the pyramid is $V=\frac{1}{3}a^2h$, which agrees with the standard formula.

Volumes of Solids of Revolution

The Method of Disks

Suppose a function f is continuous and non-negative on the interval $[a,b]$, and suppose that R is the region between the curve f and the $x-$axis (Figure 8a). If this region is revolved about the $x-$axis, it will generate a solid that will have circular cross-sections (Figure 8b) with radii of $f(x)$ at each x.

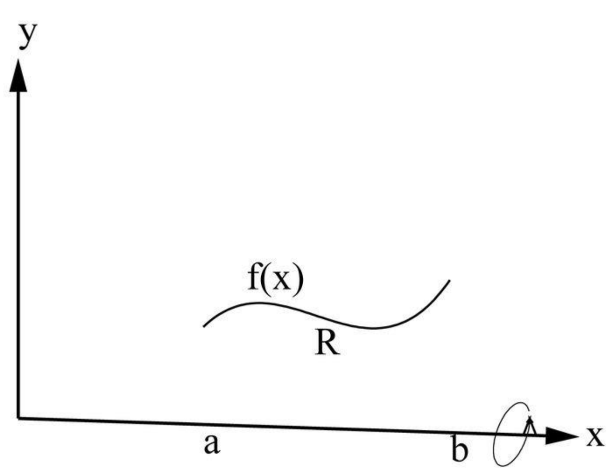

Figure 8a

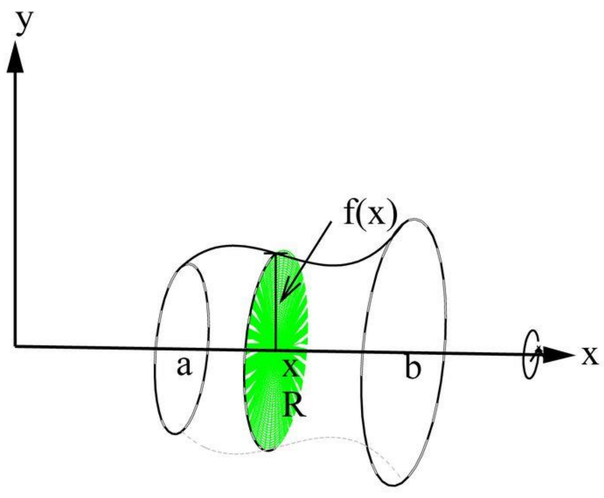

Figure 8b

Each cross-sectional area can be calculated by

$$A(x)=\pi[f(x)]^2.$$

Since the volume is defined as

$$V=\int_a^b A(x)\,dx,$$

the volume of the solid is

$$V=\int_a^b \pi[f(x)]^2\,dx.$$

Volumes by the Method of Disks (*revolution about the* x*−axis*)

$$V=\int_a^b \pi[f(x)]^2\,dx.$$

Because the shapes of the cross-sections are circular or look like the shapes of disks, the application of this method is commonly known as the ***method of disks***.

Example 2

Calculate the volume of the solid that is obtained when the region under the curve $\sqrt{x}$ is revolved about the x−axis over the interval $[1,7]$.

Solution:

As Figures 9a and 9b show, the volume is

$$V=\int_a^b \pi[f(x)]^2\,dx=\int_1^7 \pi[\sqrt{x}]^2\,dx=\pi\left[\frac{x^2}{2}\right]_1^7=24\pi.$$

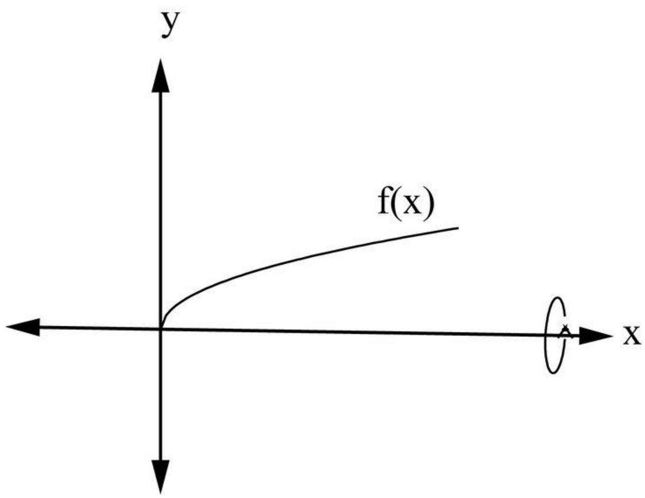

Figure 9a

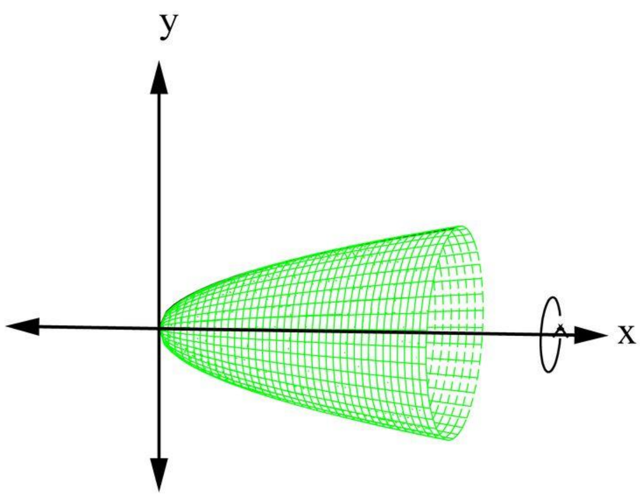

Figure 9b

Example 3:

Derive a formula for the volume of the sphere with radius r.
Solution:

One way to find the formula is to use the disk method. From your algebra, a circle of radius r and center at the origin is given by the formula
$x^2+y^2=r^2$

If we revolve the circle about the x−axis, we will get a sphere. Using the disk method, we will obtain a formula for the volume. From the equation of the circle above, we solve for y:

$f(x)=y=\sqrt{r^2-x^2}$,

thus

$$V=\int_a^b \pi[f(x)]^2dx=\int_{-r}^{+r}\pi\left[\sqrt{r^2-x^2}\right]^2dx=\pi\left[r^2x-\frac{x^3}{3}\right]_{-r}^{r}=\frac{4}{3}\pi r^3.$$

This is the standard formula for the volume of the sphere.

The Method of Washers

To generalize our results, if f and g are non-negative and continuous functions and

$f(x)\geq g(x)$

for

$a\leq x\leq b$,

Then let R be the region enclosed by the two graphs and bounded by $x=a$ and $x=b$. When this region is revolved about the x−axis, it will generate washer-like cross-sections (Figures 10a and 10b). In this case, we will have two radii: an inner radius $g(x)$ and an outer radius $f(x)$. The volume can be given by:

$$V(x)=\int_a^b \pi\left([f(x)]^2-[g(x)]^2\right)dx.$$

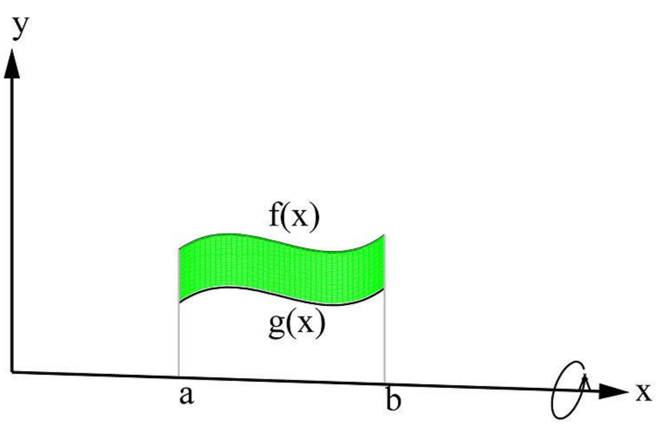

Figure 10a

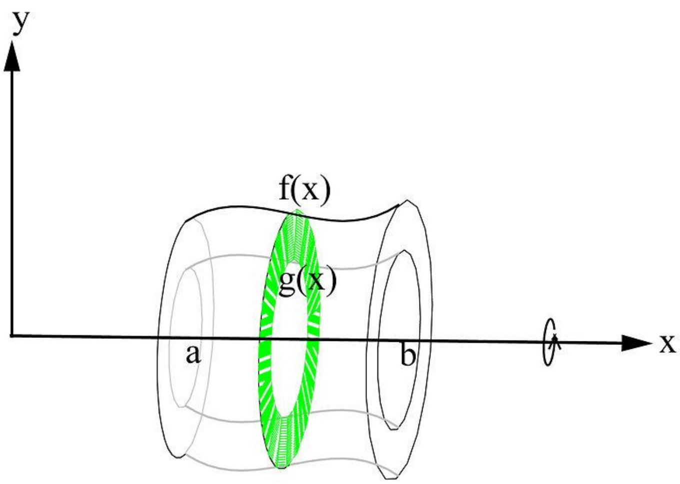

Figure 10b

Volumes by the Method of Washers (*revolution about the* $x-$*axis*)
$V(x)=\int_a^b \pi([f(x)]^2-[g(x)]^2)dx.$
Example 4:

Find the volume generated when the region between the graphs $f(x)=x^2+1$ and $g(x)=x$ over the interval $[0,3]$ is revolved about the $x-$axis.
Solution:

As Figures 11a and 11b show, the volume is

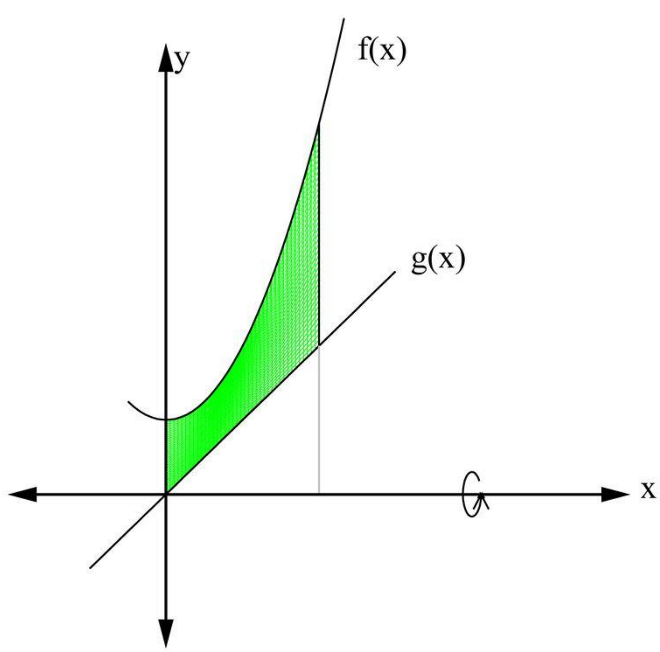

Figure 11a

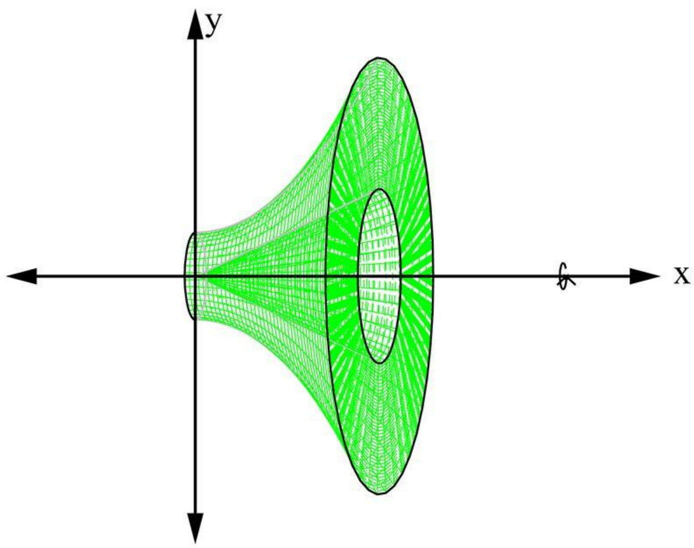

Figure 11b

From the formula above,

$V(x)=\int_a^b \pi([f(x)]^2-[g(x)]^2)dx=\int_0^3 \pi((x^2+1)^2-(x)^2)dx=\int_0^3 \pi(x^4+x^2+1)dx=\frac{303\pi}{5}.$
The methods of disks and washers can also be used if the region is revolved about the $y-$axis. The analogous formulas can be easily deduced from the above formulas or from the volumes of solids generated.

Disks:

$V=\int_c^d \pi[u(y)]^2dy.$

Washers:

$V=\int_c^d \pi([w(y)]^2-[v(y)]^2)dy.$

Example 5:

What is the volume of the solid generated when the region enclosed by $y=\sqrt{x}$, $y=3$, and $x=0$ is revolved about the $y-$axis?

Solution:

Since the solid generated is revolved about the $y-$axis (Figures 12a and 12b), we must rewrite $y=\sqrt{x}$ as $x=y^2$.
Thus $u(y)=y^2$. The volume is
$V=\int_c^d \pi[u(y)]^2dy=\int_0^3 \pi[y^2]^2dy=\int_0^3 \pi y^4dx=\pi\left[\frac{y^5}{5}\right]_0^3=\pi\left[\frac{3^5}{5}-0\right]=\frac{243\pi}{5}.$

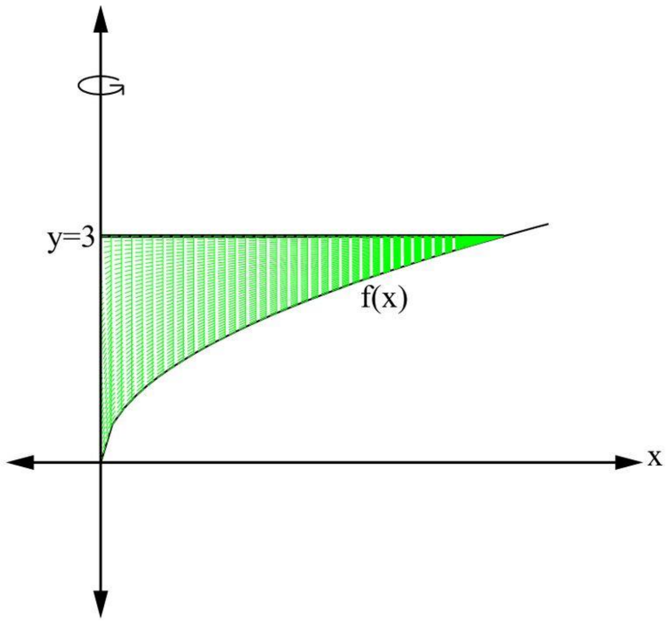

Figure 12a

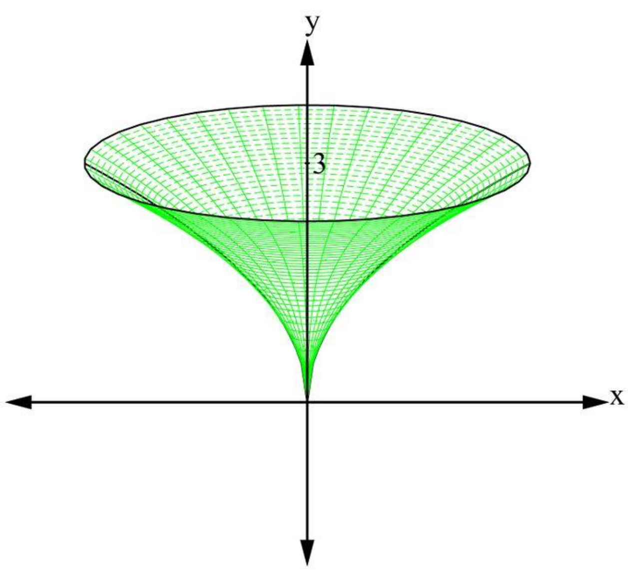

Figure 12b

Volume By Cylindrical Shells

The method of computing volumes so far depended upon computing the cross-sectional area of the solid and then integrating it across the solid. What happens when the cross-sectional area cannot be found or the integration is too difficult to solve? Here is where the ***shell method*** comes along.

To show how difficult it sometimes is to use the disk or the washer methods to compute volumes, consider the region enclosed by the function $f(x)=x-x^2$. Let us revolve it about the line $x=-1$ (Figure 13a) to generate the shape of a doughnut-shaped cake (Figure 13b). What is the volume of this solid?

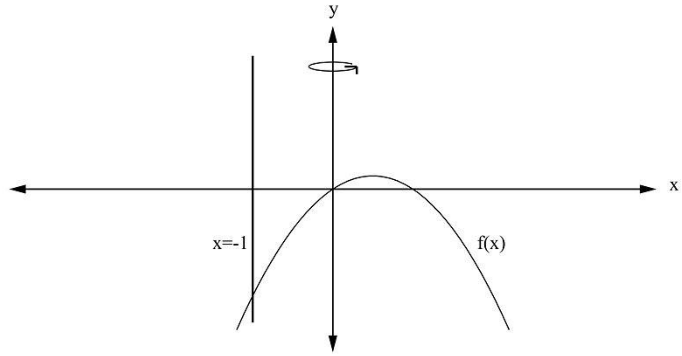

Figure 13a

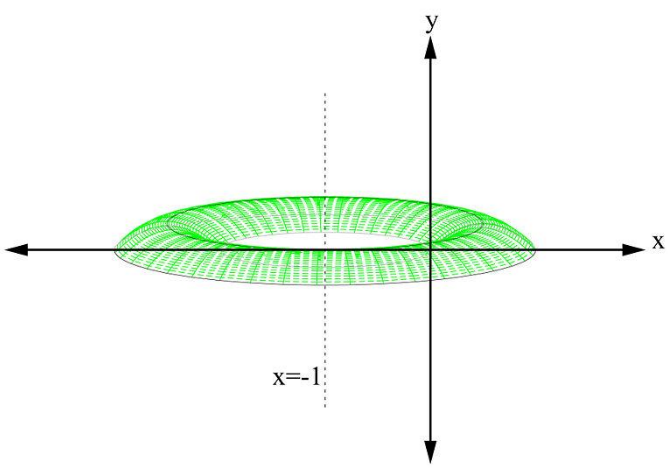

Figure 13b

If we wish to integrate with respect to the $y-$axis, we have to solve for $x$in terms of y. That would not be easy (try it!). An easier way is to integrate with respect to the $x-$axis by using the shell method. Here is how: A cylindrical shell is a solid enclosed by two concentric cylinders. If the inner radius is r_1 and the outer one is r_2, with both of height h, then the volume is (Figure 14)

$$V=[\text{area of the cross-section}]\cdot[\text{height}]=\pi(r_2^2-r_1^2)h=\pi(r_2+r_1)(r_2-r_1)h=2\pi\cdot\left[\frac{1}{2}(r_2+r_1)\right]\cdot h\cdot(r_2-r_1).$$

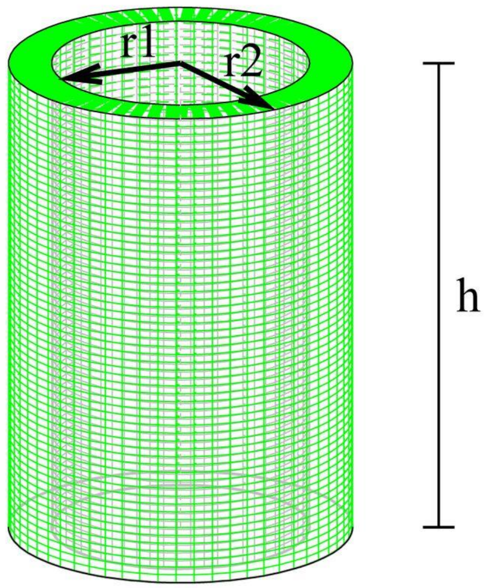

Figure 14

Notice however that (r_2-r_1) is the thickness of the shell and $\frac{1}{2}(r_2+r_1)$ is the average radius of the shell.
Thus

$$V=2\pi\cdot[\text{average radius}]\cdot[\text{height}]\cdot[\text{thickness}].$$

Replacing the average radius with a single variable r and using h for the height, we have

$$V=2\pi\cdot r\cdot h\cdot[\text{thickness}].$$

In general the shell's thickness will be dx or dy depending on the axis of revolution. This discussion leads to the following formulas for rotation about an axis. We will then use this formula to compute the volume V of the solid of revolution that is generated by revolving the region about the $x-$axis.

Volume By Cylindrical Shell about the $y-$Axis

Suppose f is a continuous function in the interval $[a,b]$ and the region $R$is bounded above by $y=f(x)$ and below by the $x-$axis, and on the sides by the lines $x=a$ and $x=b$. If R is rotated around the $y-$axis, then the cylinders are vertical, with $r=x$ and $h=f(x)$. The volume of the solid is given by

$$V=\int_a^b 2\pi rh\,dx=\int_a^b 2\pi x f(x)\,dx.$$

Volume By Cylindrical Shell about the $x-$Axis

Equivalently, if the volume is generated by revolving the same region about the $x-$axis, then the cylinders are horizontal with

$v=\int_c^d 2\pi rh\,dy$,

where $c=f^{-1}(a)$ and $d=f^{-1}(b)$. The values of r and h are determined in context, as you will see in Example 6.

Note: Example 7 shows what to do when the rotation is not about an axis.

Example 6:

A solid figure is created by rotating the region R (Figure 15) around the $x-$axis. R is bounded by the curve $y=x^2$ and the lines $x=0$ and $x=2$. Use the shell method to compute the volume of the solid.

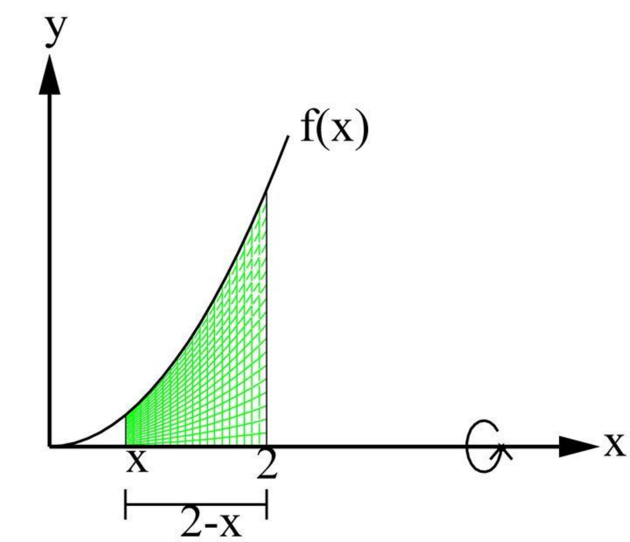

Figure 15

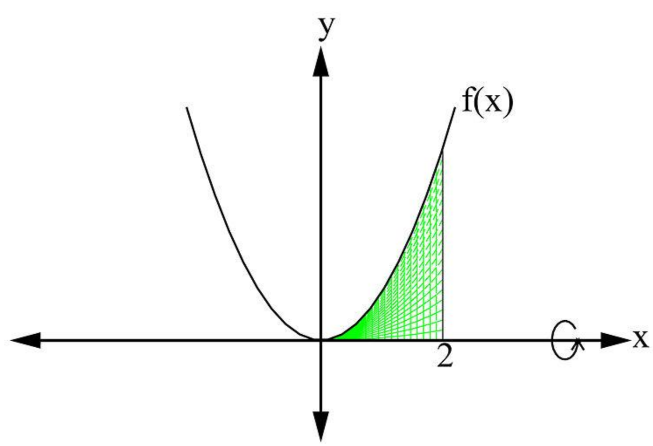

Figure 16

Solution:

From Figure 15 we can identify the limits of integration: y runs from 0 to 4. A horizontal strip of this region would generate a cylinder with height $2-y\surd$ and radius y. Thus the volume of the solid will be

$V=\int_{c}^{d}2\pi rhdy=\int_{0}^{4}2\pi y(2-y\surd)dy=2\pi\int_{0}^{4}(2y-y^{3/2})dy=2\pi[y^2-25y^{5/2}]_{0}^{4}=32\pi5.$

Note: The alert reader will have noticed that this example could be worked with a simpler integral using disks. However, the following example can only be solved with shells.

Example 7:

Find the volume of the solid generated by revolving the region bounded by $y=x^3+12x+14$, $y=14$, and $x=1$, about $x=3$.

Solution:

As you can see, the equation $y=x^3+12x+14$ cannot be easily solved for x and therefore it will be necessary to solve the problem by the shell method. We are revolving the region about a line parallel to the $y-$axis and thus integrate with respect to x. Our formula is

$V=\int_{a}^{b}2\pi rhdx.$

In this case, the radius is $3-x$ and the height is $x^3+12x+14-14$. Substituting,

$V=2\pi\int_{0}^{1}(3-x)(x^3+12x+14-14)dx=2\pi\int_{0}^{1}(-x^4+3x^3-12x^2+32x)dx=2\pi[-15x^5+34x^4-16x^3+34x^2]_{0}^{1}=2\pi[-15+34-16+34]=2\pi[1715]=34\pi15.$

Review Questions

In problems #1 - 4, find the volume of the solid generated by revolving the region bounded by the curves about the $x-$axis.

1. $y=9-x^2-----\surd,y=0$
2. $y=3+x,y=1+x^2$
3. $y=\sec x,y=2-\surd,-\pi/4\le x\le\pi/4$
4. $y=1,y=x,x=0$

In problems #5–8, find the volume of the solid generated by revolving the region bounded by the curves about the $y-$axis.

5. $y=x^3,x=0,y=1$
6. $x=y^2,y=x-2$
7. $x=\csc y,y=\pi/4,y=3\pi/4,x=0$
8. $y=0,y=x--\surd,x=4$

In problems #9–12, use cylindrical shells to find the volume generated when the region bounded by the curves is revolved about the axis indicated.

9. $y=1x, y=0, x=1, x=3$, about the $y-$axis
10. $y=x^2, x=1, y=0$, about the $x-$axis
11. $y=2x-1, y=-2x+3, x=2$, about the $y-$axis
12. $y^2=x, y=1, x=0$, about the $x-$axis.
13. Use the cylindrical shells method to find the volume generated when the region is bounded by $y=x^3, y=1, x=0$ is revolved about the line $y=1$.