Learning Objectives

A student will be able to:

- Compute the area between two curves with respect to the $x-$ and $y-$axes.

In the last chapter, we introduced the definite integral to find the area between a curve and the $x-$ axis over an interval $[a,b]$. In this lesson, we will show how to calculate the area between two curves.

Consider the region bounded by the graphs f and g between $x=a$and $x=b$, as shown in the figures below. If the two graphs lie above the $x-$axis, we can interpret the area that is sandwiched between them as the area under the graph of g subtracted from the area under the graph f.

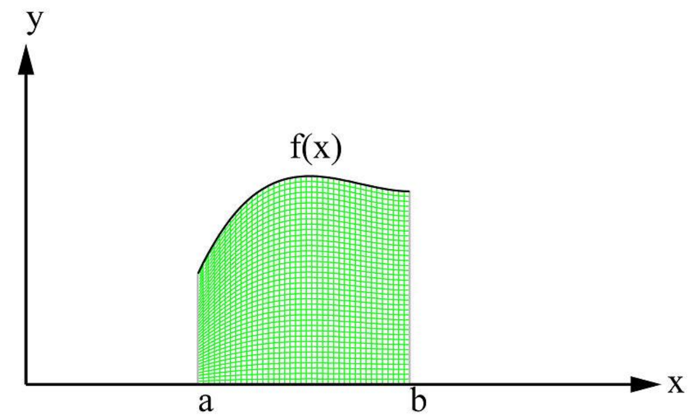

Figure 1a

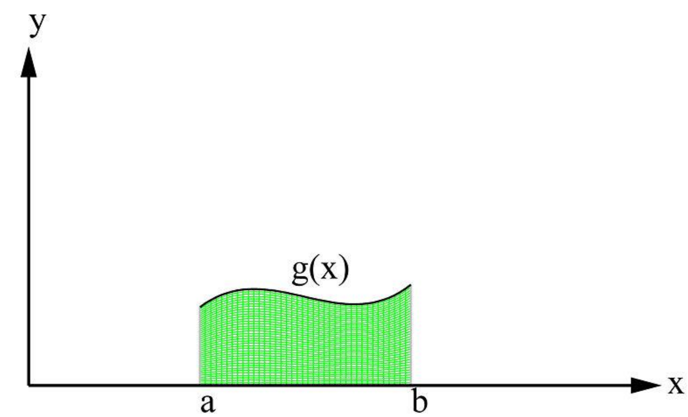

Figure 1b

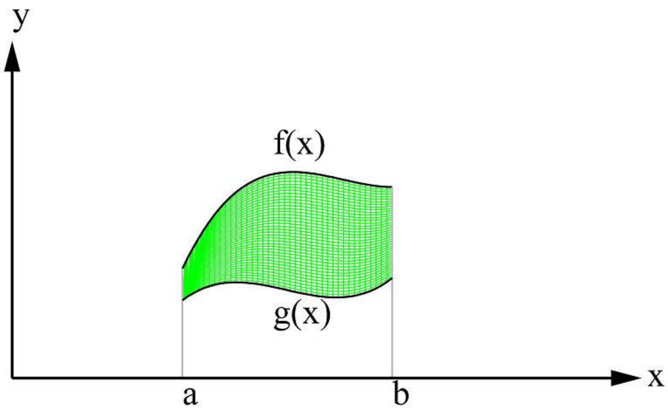

Figure 1c

Therefore, as the graphs show, it makes sense to say that

[Area under f (Fig. 1a)] − [Area under g (Fig. 1b)] = [Area between $f$and g (Fig. 1c)], $\int_a^b f(x)dx - \int_a^b g(x) = \int_a^b [f(x) - g(x)]dx.$
This relation is valid as long as the two functions are continuous and the upper function $f(x) \geq g(x)$ on the interval $[a,b]$.

The Area Between Two Curves (*With respect to the* $x-$*axis*)

If f and g are two continuous functions on the interval $[a,b]$ and $f(x) \geq g(x)$ for all values of x in the interval, then the area of the region that is bounded by the two functions is given by
$A = \int_a^b [f(x) - g(x)]dx.$

Example 1:

Find the area of the region enclosed between $y=x^2$ and $y=x+6$.

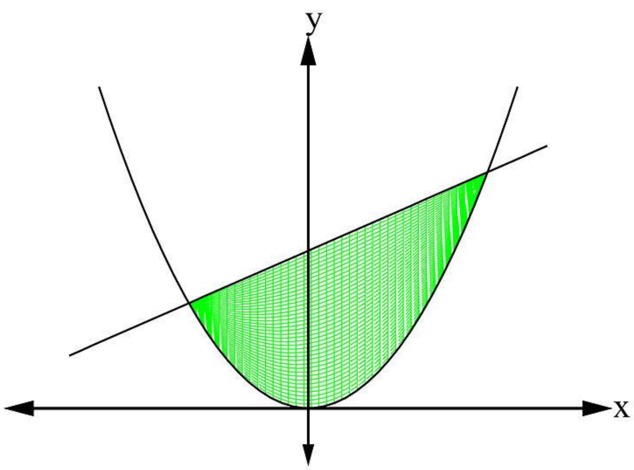

Figure 2

Solution:

We first make a sketch of the region (Figure 2) and find the end points of the region. To do so, we simply equate the two functions,

$x^2=x+6$,
and then solve for x.
$x^2-x-6(x+2)(x-3)=0=0$
from which we get $x=-2$ and $x=3$.
So the upper and lower boundaries intersect at points $(-2,4)$ and $(3,9)$.
As you can see from the graph, $x+6 \geq x^2$ and hence $f(x)=x+6$and $g(x)=x^2$ in the interval $[-2,3]$. Applying the area formula,
$A=\int_a^b[f(x)-g(x)]dx=\int_{-2}^{3}[(x+6)-(x^2)]dx$.
Integrating,

$A=[\frac{x^2}{2}+6x-\frac{x^3}{3}]_{-2}^{3}=\frac{125}{6}$.
So the area between the two curves $f(x)=x+6$ and $g(x)=x^2$ is $125/6$.
Sometimes it is possible to apply the area formula with respect to the $y-$coordinates instead of the $x-$coordinates. In this case, the equations of the boundaries will be written in such a way that y is expressed explicitly as a function of x (Figure 3).

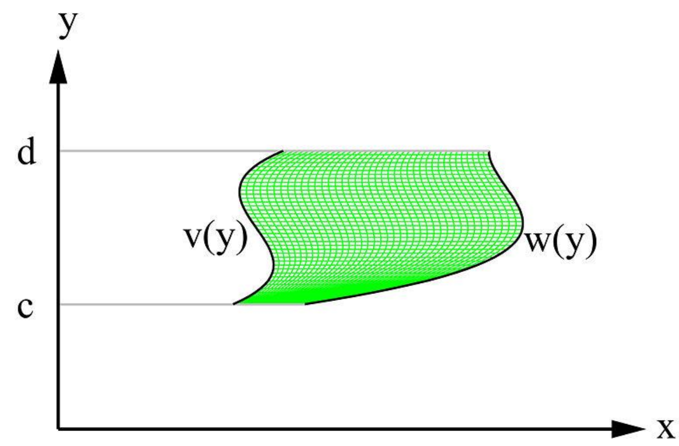

Figure 3

The Area Between Two Curves (*With respect to the* $y-$*axis*)
If w and v are two continuous functions on the interval $[c,d]$ and $w(y) \geq v(y)$ for all values of y in the interval, then the area of the region that is bounded by $x=v(y)$ on the left, $x=w(y)$ on the right, below by $y=c$, and above by $y=d$, is given by
$A=\int_c^d[w(y)-v(y)]dy$.
Example 2:

Find the area of the region enclosed by $x=y^2$ and $y=x-6$.
Solution:

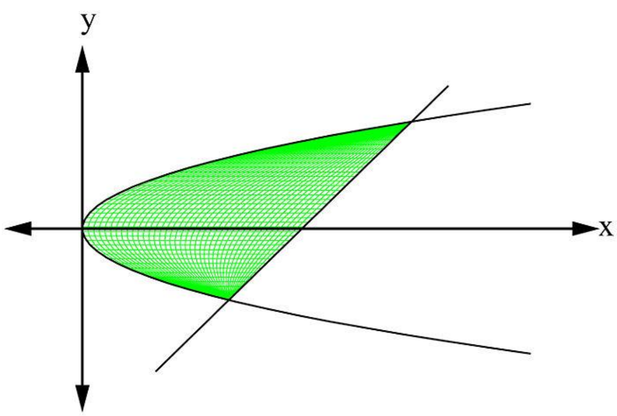

Figure 4

As you can see from Figure 4, the left boundary is $x=y^2$ and the right boundary is $y=x-6$. The region extends over the interval $-2\le y\le 3$. However, we must express the equations in terms of y.We rewrite
$xx=y^2=y+6$
Thus

$$A=\int_{-2}^{3}[y+6-y^2]dy=\left[\frac{y^2}{2}+6y-\frac{y^3}{3}\right]_{-2}^{3}=\frac{125}{6}.$$

Review Questions

In problems #1 - 7, sketch the region enclosed by the curves and find the area.

1. $y=x^2, y=\sqrt{x}$, on the interval $[0.25,1]$
2. $y=0$, $y=\cos 2x$, on the interval $[\frac{\pi}{4},\frac{\pi}{2}]$
3. $y=|-1+x|+2$, $y=-15x+7$
4. $y=\cos x$, $y=\sin x$, $x=0$, $x=2\pi$
5. $x=y^2$, $y=x-2$, integrate with respect to y
6. $y^2-4x=4$, $4x-y=16$ integrate with respect to y
7. $y=8\cos x$, $y=\sec^2 x$, $-\pi/3\le x\le \pi/3$
8. Find the area enclosed by $x=y^3$ and $x=y$.
9. If the area enclosed by the two functions $y=k\cos x$ and $y=kx^2$ is 2, what is the value of k?
10. Find the horizontal line $y=k$ that divides the region between $y=x^2$and $y=9$ into two equal areas.