## Learning Objectives

- Integrate composite functions
- Use change of variables to evaluate definite integrals
- Use substitution to compute definite integrals

## Introduction

In this lesson we will expand our methods for evaluating definite integrals. We first look at a couple of situations where finding antiderivatives requires special methods. These involve finding antiderivatives of composite functions and finding antiderivatives of products of functions.

***Antiderivatives of Composites***

Suppose we needed to compute the following integral:

$$\int 3x^2\sqrt{1+x^3}\,dx.$$

Our rules of integration are of no help here. We note that the integrand is of the form $f(g(x))*g'(x)$ where $g(x)=1+x^3$ and $f(x)=\sqrt{x}$.
Since we are looking for an antiderivative $F$ of $f$, and we know that $F'=f$, we can re-write our integral as

$$\int \sqrt{1+x^3}\cdot 3x^2\,dx=\frac{2}{3}\left(\sqrt{1+x^3}\right)^{\frac{3}{2}}+C.$$

In practice, we use the following substitution scheme to verify that we can integrate in this way:

1. Let $u=1+x^3$.
2. Differentiate both sides so $du=3x^2dx$.
3. Change the original integral in $x$ to an integral in $u$:

   $\int \sqrt{1+x^3}\cdot 3x^2\,dx=\int \sqrt{u}\,du$, where $u=1+x^3$ and $du=3x^2dx$.

4. Integrate with respect to $u$:

$$\int \sqrt{u}\,du=\int u^{\frac{1}{2}}\,du=\frac{2}{3}u^{\frac{3}{2}}+C.$$

5. Change the answer back to $x$:

$$\int \sqrt{u}\,du=\frac{2}{3}u^{\frac{3}{2}}+C=\frac{2}{3}\left(\sqrt{1+x^3}\right)^{\frac{3}{2}}+C.$$

While this method of substitution is a very powerful method for solving a variety of problems, we will find that we sometimes will need to modify the method slightly to address problems, as in the following example.

**Example 1:**

Compute the following indefinite integral:

$\int x^2 e^{x^3} dx.$

**Solution:**

We note that the derivative of $x^3$ is $3x^2$; hence, the current problem is not of the form $\int F'(g(x))\cdot g'(x)dx$. But we notice that the derivative is off only by a constant of $3$ and we know that constants are easy to deal with when differentiating and integrating. Hence

Let $u=x^3$.

Then $du=3x^2dx$.

Then $\frac{1}{3}du=x^2dx$. and we are ready to change the original integral from $x$ to an integral in $u$ and integrate:

$\int x^2e^{x^3}dx=\int e^u\left(\frac{1}{3}du\right)=\frac{1}{3}\int e^u du=\frac{1}{3}e^u+C.$

Changing back to $x$, we have

$\int x^2e^{x^3}dx=\frac{1}{3}e^{x^3}+C.$

We can also use this substitution method to evaluate definite integrals. If we attach limits of integration to our first example, we could have a problem such as

$\int_1^4 \sqrt{1+x^3}\cdot 3x^2dx.$

The method still works. However, we have a choice to make once we are ready to use the Fundamental Theorem to evaluate the integral.

Recall that we found that $\int \sqrt{1+x^3}\cdot 3x^2dx=\int \sqrt{u}\,du$ for the indefinite integral. At this point, we could evaluate the integral by changing the answer back to $x$ or we could evaluate the integral in $u$.But we need to be careful. Since the original limits of integration were in $x$, we need to change the limits of integration for the equivalent integral in $u$. Hence,

$\int_1^4 \sqrt{1+x^3}\cdot 3x^2dx=\int_{u=2}^{65}\sqrt{u\,du}$, where $u=1+x^3$

$\int_1^4 \sqrt{1+x^3}\cdot 3x^2dx=\int_{u=2}^{65}\sqrt{u\,du}=\frac{2}{3}u^{\frac{3}{2}}\Big|_{u=2}^{u=65}=\frac{2}{3}(\sqrt{65^3}-\sqrt{8}).$

***Integrating Products of Functions***

We are not able to state a rule for integrating products of functions, $\int f(x)g(x)dx$ but we can get a relationship that is almost as effective. Recall how we differentiated a product of functions:

$\frac{d}{dx}f(x)g(x)=f(x)g'(x)+g(x)f'(x).$

So by integrating both sides we get

$\int[f(x)g'(x)+g(x)f'(x)]dx=f(x)g(x)$, or

$\int f(x)g'(x)dx=f(x)g(x)-\int g(x)f'(x).$

In order to remember the formula, we usually write it as

$$\int u\,dv = uv - \int v\,du.$$

We refer to this method as integration by parts. The following example illustrates its use.

**Example 2:**

Use integration by parts method to compute

$$\int x e^x dx.$$

**Solution:**

We note that our other substitution method is not applicable here. But our integration by parts method will enable us to reduce the integral down to one that we can easily evaluate.

Let $u=x$ and $dv=e^x dx$ then $du=dx$ and $v=e^x$

By substitution, we have

$$\int x e^x dx = x e^x - \int e^x dx.$$

We can easily evaluate the integral and have

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + C.$$

And should we wish to evaluate definite integrals, we need only to apply the Fundamental Theorem to the antiderivative.

## Lesson Summary

1. We integrated composite functions.
2. We used change of variables to evaluate definite integrals.
3. We used substitution to compute definite integrals.

## Review Questions

Compute the integrals in problems #1–10.

1. $\int x \ln x\, dx$
2. $\int_{31} x--\sqrt{}\ln x\, dx$
3. $\int x 2x+1-----\sqrt{}dx$
4. $\int_{10} x^3 1-x^2-----\sqrt{}dx$
5. $\int x \cos x\, dx$
6. $\int_{10} x^2 x^3+9-----\sqrt{}dx$
7. $\int (1x^2 \cdot e^{1x})dx$
8. $\int x^3 e^{x^2} dx$

9. $\int \ln x x_{52} dx$
10. $\int_{e1} 1xdx$