Learning Objectives

- Use the Fundamental Theorem of Calculus to evaluate definite integrals

Introduction

In the Lesson on Evaluating Definite Integrals, we evaluated definite integrals using antiderivatives. This process was much more efficient than using the limit definition. In this lesson we will state the Fundamental Theorem of Calculus and continue to work on methods for computing definite integrals.

Fundamental Theorem of Calculus:

Let f be continuous on the closed interval $[a,b]$.

1. If function F is defined by $F(x)=\int_a^x f(t)dt$, on $[a,b]$, then $F'(x)=f(x)$ on $[a,b]$.
2. If g is any antiderivative of f on $[a,b]$, then

$\int_a^b f(t)dt=g(b)-g(a)$.

We first note that we have already proven part 2 as Theorem 4.1. The proof of part 1 appears at the end of this lesson.

Think about this Theorem. Two of the major unsolved problems in science and mathematics turned out to be solved by calculus which was invented in the seventeenth century. These are the ancient problems:

1. Find the areas defined by curves, such as circles or parabolas.
2. Determine an instantaneous rate of change or the slope of a curve at a point.

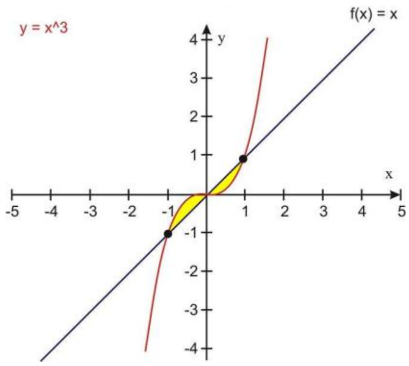
With the discovery of calculus, science and mathematics took huge leaps, and we can trace the advances of the space age directly to this Theorem.

Let's continue to develop our strategies for computing definite integrals. We will illustrate how to solve the problem of finding the area bounded by two or more curves.

Example 1:

Find the area between the curves of $f(x)=x$ and $g(x)=x^3$. for $-1\leq x\leq 1$.
Solution:

We first observe that there are no limits of integration explicitly stated here. Hence we need to find the limits by analyzing the graph of the functions.



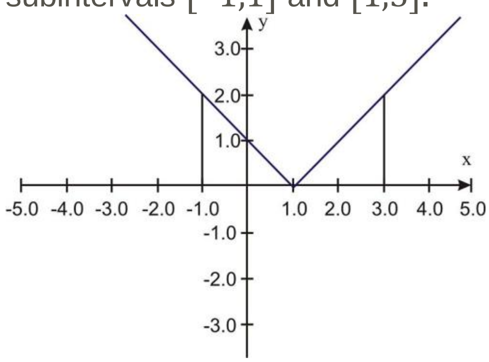
We observe that the regions of interest are in the first and third quadrants from $x=-1$ to $x=1$. We also observe the symmetry of the graphs about the origin. From this we see that the total area enclosed is $2\int_0^1(x-x^3)dx=2\left[\int_0^1 x\,dx-\int_0^1 x^3dx\right]=2\left[\frac{x^2}{2}\Big|_0^1-\frac{x^4}{4}\Big|_0^1\right]=2\left[\frac{1}{2}-\frac{1}{4}\right]=2\left[\frac{1}{4}\right]=\frac{1}{2}$.

Example 2:

Find the area between the curves of $f(x)=|x-1|$ and the x−axis from $x=-1$ to $x=3$.

Solution:

We observe from the graph that we will have to divide the interval $[-1,3]$into subintervals $[-1,1]$ and $[1,3]$.



Hence the area is given by

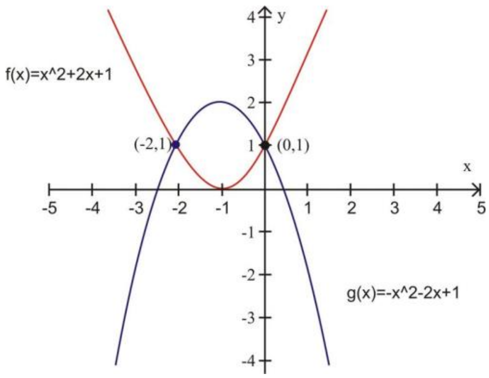
$\int_{-1}^{1}(-x+1)dx+\int_1^3(x-1)dx=\left(-\frac{x^2}{2}+x\right)\Big|_{-1}^{+1}+\left(\frac{x^2}{2}-x\right)\Big|_{+1}^{+3}=2+2=4$.

Example 3:

Find the area enclosed by the curves of $f(x)=x^2+2x+1$ and $g(x)=-x^2-2x+1$.

Solution:

The graph indicates the area we need to focus on.



$$\int_{-2}^{0}(-x^2-2x+1)dx-\int_{-2}^{0}(x^2+2x+1)dx=\left(-\frac{x^3}{3}-x^2+x\right)\Big|_{-2}^{0}-\left(\frac{x^3}{3}+x^2+x\right)\Big|_{-2}^{0}=\frac{8}{3}.$$

Before providing another example, let's look back at the first part of the Fundamental Theorem. If function F is defined by $F(x)=\int_a^x f(t)dt,$ on $[a,b]$ then $F'(x)=f(x)$ on $[a,b]$. Observe that if we differentiate the integral with respect to x, we have

$$\frac{d}{dx}\int_a^x f(t)dt=F'(x)=f(x).$$

This fact enables us to compute derivatives of integrals as in the following example.

Example 4:

Use the Fundamental Theorem to find the derivative of the following function:

$$g(x)=\int_0^x (1+\sqrt[3]{t})dt.$$

Solution:

While we could easily integrate the right side and then differentiate, the Fundamental Theorem enables us to find the answer very routinely.

$$g'(x)=\frac{d}{dx}\int_0^x (1+\sqrt[3]{t})dt=1+\sqrt[3]{x}.$$

This application of the Fundamental Theorem becomes more important as we encounter functions that may be more difficult to integrate such as the following example.

Example 5:

Use the Fundamental Theorem to find the derivative of the following function:

$$g(x)=\int_2^x (t^2\cos t)dt.$$

Solution:

In this example, the integral is more difficult to evaluate. The Fundamental Theorem enables us to find the answer routinely.

$g'(x)=\frac{d}{dx}\int_2^x (t^2\cos t)dt=x^2\cos x.$

Lesson Summary

1. We used the Fundamental Theorem of Calculus to evaluate definite integrals.

Fundamental Theorem of Calculus

Let f be continuous on the closed interval $[a,b]$.

1. If function F is defined by $F(x)=\int_a^x f(t)dt$, on $[a,b]$, then $F'(x)=f(x)$, on $[a,b]$.
2. If g is any antiderivative of f on $[a,b]$, then

$\int_a^b f(t)dt=g(b)-g(a).$

We first note that we have already proven part 2 as Theorem 4.1.

Proof of Part 1.

1. Consider $F(x)=\int_a^x f(t)dt$, on $[a,b]$.
2. $x,c\in[a,b]$, $c<x$.

Then $\int_a^x f(t)dt=\int_a^c f(t)dt+\int_c^x f(t)dt$ by our rules for definite integrals.

3. Then $\int_a^x f(t)dt-\int_a^c f(t)dt=\int_c^x f(t)dt$. Hence $F(x)-F(c)=\int_c^x f(t)dt$.
4. Since f is continuous on $[a,b]$ and $x,c\in[a,b]$, $c<x$ then we can select $u,v\in[c,v]$ such that $f(u)$ is the minimum value of and $f(v)$ is the maximum value of f in $[c,x]$. Then we can consider $f(u)(x-c)$ as a lower sum and $f(v)(x-c)$ as an upper sum of f from c to x. Hence
5. $f(u)(x-c)\le\int_c^x f(t)dt\le f(v)(x-c)$.
6. By substitution, we have:

$f(u)(x-c)\le F(x)-F(c)\le f(v)(x-c).$

7. By division, we have

$f(u)\le\frac{F(x)-F(c)}{x-c}\le f(v).$

8. When x is close to c, then both $f(u)$ and $f(v)$ are close to $f(c)$ by the continuity of f
9. Hence $\lim_{x\to c^+}\frac{F(x)-F(c)}{x-c}=f(c)$. Similarly,

if $x<c$, then $\lim_{x\to c^-}\frac{F(x)-F(c)}{x-c}=f(c)$. Hence, $\lim_{x\to c}\frac{F(x)-F(c)}{x-c}=f(c)$.

10. By the definition of the derivative, we have that

$F'(c)=\lim_{x\to c}\frac{F(x)-F(c)}{x-c}=f(c)$ for every $c\in[a,b]$. Thus, F is an antiderivative of f on $[a,b]$.

Review Questions

In problems #1–4, sketch the graph of the function $f(x)$ in the interval $[a,b]$. Then use the Fundamental Theorem of Calculus to find the area of the region bounded by the graph and the $x-$axis.

1. $f(x)=2x+3,\ [0,4]$
2. $f(x)=e^x,\ [0,2]$
3. $f(x)=x^2+x,\ [1,3]$
4. $f(x)=x^2-x,\ [0,2]$

(Hint: Examine the graph of the function and divide the interval accordingly.)

In problems #5–7 use antiderivatives to compute the definite integral.

5. $\int_{-1}^{+1}|x|dx$
6. $\int_{0}^{3}|x^3-2|dx$

(Hint: Examine the graph of the function and divide the interval accordingly.)

7. $\int_{-2}^{+4}[|x-1|+|x+1|]dx$

(Hint: Examine the graph of the function and divide the interval accordingly.)

In problems #8–10, find the area between the graphs of the functions.

8. $f(x)=\sqrt{x},\ g(x)=x,\ [0,2]$
9. $f(x)=x^2,\ g(x)=4,\ [0,2]$
10. $f(x)=x^2+1,\ g(x)=3-x,\ [0,3]$