

CK-12 Texas Instruments Calculus Student Edition

Lori Jordan

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Contents

1	SE Introduction - TI	1
1.1	Calculus TI Resources	2
2	SE Functions, Limits, and Continuity	3
2.1	Modeling Data	4
2.2	What is Calculus?	6
2.3	One-Sided Limits	8
3	SE Differentiation - TI	11
3.1	Getting' the Swing	12
3.2	Move those Chains	14
3.3	Implicit Differentiation	16
4	SE Applications of Derivatives - TI	19
4.1	Xtreme Calculus	20
4.2	Helicopter Bungee Jump	22
4.3	Optimization	24
4.4	Linear Approximations	26
5	SE Integration - TI	29
5.1	Exploring Area under the Curve	30
5.2	Sum Rectangles	33
5.3	FTC Changed History	35
6	SE Applications of Integration - TI	38
6.1	The Area Between Curves	39
6.2	Volume by Cross-Sections	41
6.3	Gateway Arc Length	43
7	SE Transcendental Functions - TI	46
7.1	Inverses of Functions	47
7.2	The Logarithmic Derivative	50
8	SE Integration Techniques - TI	53
8.1	Integration by Substitution	54
8.2	Integration by Parts	56
8.3	Charged Up	59
9	SE Infinite Series - TI	61
9.1	Exploring Geometric Sequences	62
9.2	Infinite Geometric Series	65
9.3	Mr. Taylor, I Presume?	68
9.4	AP Calculus Exam Prep	71

CHAPTER **1**

SE Introduction - TI

Chapter Outline

1.1 **CALCULUS TI RESOURCES**

1.1 Calculus TI Resources

Student Edition

Introduction

This FlexBook® resource contains Texas Instruments (TI) Resources for the **TI-89**, unless otherwise indicated. All the activities in this FlexBook® resource supplement the lessons in our Calculus student edition text. Teachers may need to download programs from www.timath.com that will implement or assist in the activities. All activities are listed in the same order as the Teacher's Edition.

There are also corresponding links in the student edition of CK-12 Calculus FlexBook® textbook.

- CK-12 Calculus: <http://www.ck12.org/book/CK-12-Calculus/>

CHAPTER

2**SE Functions, Limits, and
Continuity****Chapter Outline**

- 2.1 MODELING DATA**
 - 2.2 WHAT IS CALCULUS?**
 - 2.3 ONE-SIDED LIMITS**
-

The activities below are intended to supplement our CK-12 Calculus FlexBook® textbook, [Chapter 1](#) .

2.1 Modeling Data

This activity is intended to supplement Calculus, Chapter 1, Lesson 3.

Problem 1 –Exponential Growth

When Connor was born, his parents put \$1,000 dollars into an account to give him as a present on his 21st birthday. However, his parents forgot the yearly interest rate on the account.






Enter the data to the right in $L1$ and $L2$ by pressing **STAT** **ENTER**.

L1	L2	L3	1
0	1000	5	
1	1040	25	
2	1081.6	50	
3	1124.9	150	
4	1169.9	250	
5	1216.7	500	
6	1265.3	-----	

L1(1) = 0

Create a scatter plot of the data by pressing 2^{nd} [$Y =$] **ENTER** and matching the screen to the right.

To view the scatter plot, press **ZOOM** and select **9:ZoomStat**.

2103	Plot2	Plot3
On	Off	
Type:		
		
Xlist:	L1	
Ylist:	L2	
Mark:		+ .

Using your knowledge of compound interest, study the data and the graph to determine a function for the growth of Connor's money.

Enter your equation in $Y1$ and press **GRAPH** to check your result.

Note: the regressions can be found by pressing **STAT** and scrolling over to the **CALC** menu. Select **ExpReg**, **ENTER** 2^{nd} [1] [,] 2^{nd} [2] **ENTER**.

The equation for the data is: _____

Problem 2 –Logarithmic Growth

Scientists are testing the amount of greenhouse gases present at a research site near the north pole to determine the effect on polar ice melting. Enter in the results in $L3$ and $L4$ of your graphing calculator.

L3	L4	L5	3
5	.59432	-----	
25	1.1886		
50	1.4446		
150	1.8503		
250	2.0389		
500	2.2949		
-----	-----		
L3(1)=5			

Create a scatter plot of the data by pressing 2^{nd} [Y =], selecting **1:Plot1**, and matching the screen to the right.

To view the scatter plot, press **ZOOM** and select **9:ZoomStat**.

Note: $L3$ and $L4$ are found by pressing 2^{nd} [3] and 2^{nd} [4] respectively.

Determine a natural log equation to model the data by pressing **STAT**, scrolling over to **CALC** and selecting **9:LnReg**.

To finish the regression, enter, $L3, L4, Y1$ so the screen reads **LnReg $L3, L4, Y1$**

Press **GRAPH** to view the scatter plot and regression equation both plotted.

The equation for the data is: _____

Extension –Exponential Decay

You will need to download L5.8xl and L6.8xl from your teacher for this extension.

Due to an environmental spill, a farmer is losing the amount of land on which he can plant crops. The data in $L5$ and $L6$ of your graphing calculator show the year and amount of useable land for each year. Determine an equation that models the amount of land the farmer can use each year.

After creating a scatter plot of the data, use your graphing calculator to perform an exponential regression that models the data and plot the function by pressing **GRAPH**.

The equation for the data is: _____

2.2 What is Calculus?

This activity is intended to supplement Calculus, Chapter 1, Lesson 4.

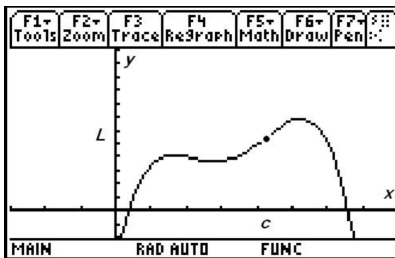
Part 1 –Introduction & Limits

1. Before taking calculus, what had you heard about the subject?

Calculus comes from a Latin word for a stone used for counting. Calculus deals with counting infinitesimals (something *really* small).

Simply put, this highly applicable discipline of mathematics is about **slope** and **area**. The concept of limit defines these two major calculus topics of the derivative and integral.

2. View the graph to the right. As x approaches c , what is the value the function is approaching? In other words, as x gets closer to c , but not equal to c , the value of $f(x)$ gets closer to what one value?



Part 2 –Derivatives & Optimization

Extrema points are the maxima and minima points on a curve. These extreme values could just be highest and lowest points in a neighborhood, or local region.

Run the program *whatcalc* by typing **whatcalc()** on the Home screen and pressing **ENTER**. When prompted, select **1:Graph 1**.

3. For the 4th degree polynomial shown, how many extrema does this function have?

4. What is the slope of the tangent at these local maximum and minimum points?

The **derivative**, or slope of the tangent line, is the limit of the slope of secant lines. Now select **2:Graph 2** from the program drop down menu.

The derivative is useful to solve all sorts of problems involving rates of change in medicine, economics, science, and engineering. Calculus can help a manufacturer find the optimal shape to minimize cost. Optimization of functions is extremely beneficial for real-world problems.

For example, a manufacturer may want to maximize the volume of a box given a certain sized piece of cardboard. The box can be formed by folding or cutting the corners. Optimizing the volume would help determine the dimension of the box with the largest volume. The next graph you will view represents the volume of the box. Setting the derivative of the function equal to zero would give the exact answer.

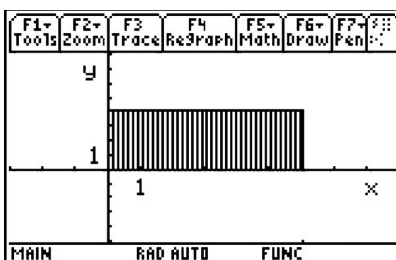
Select **3:Graph 3** from the program drop down menu.

5. a. Trace the graph to approximate maximum volume. What is the maximum and when does it occur?
- b. Using calculus, how could the exact maximum value of the volume function be found?

Part 3 –Integrals and History

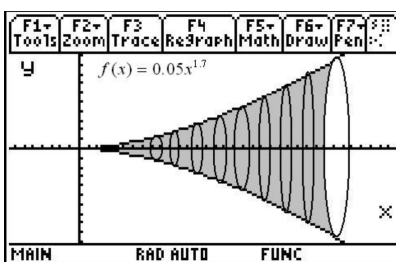
Adding rectangles will come up in the study of calculus.

6. What is the area of the rectangle shown to the right?



Select **4:Graph 4** from the program menu. The volume can be found by adding cylinders of width dx .

7. What line is this function rotated about?



Calculus was independently co-discovered by Sir Isaac **Newton** (in England) and Gottfried Wilhelm **Leibniz** (in Germany). Newton actually was the first to discover the inverse relationship between slope and area—the derivative and integral—but Leibniz published first. This relationship is called the **Fundamental Theorem of Calculus**.

Newton made use of calculus in his book *Principia*, the greatest scientific work of all time and the work which marks the beginning of the Scientific Revolution.

The creation of nearly all technology today could be traced back to the Fundamental Theorem of Calculus.

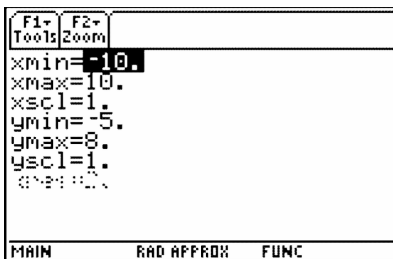
8. Who were the co-discoverers of calculus (even though they did not work together)?
9. What did you learn from this activity? Especially explain your current understanding of the three main calculus concepts.

2.3 One-Sided Limits

This activity is intended to supplement Calculus, Chapter 1, Lesson 6.

Set up –graphing piecewise functions that show discontinuity

1. After turning on your device, go to the $Y =$ screen by pressing $[\blacklozenge]F1$.
 2. Turn the functions off or clear them; press $F1 >$ **Clear Functions**.
- Note: You can turn functions off by un-checking them using $F4$.
3. Turn Discontinuity Detection on. Press $F1 >$ **Format** to find the option for Discontinuity Detection.
 4. Set the window, using $[\blacklozenge]F2$, to the settings shown at the right.
 5. Back on the $Y =$ screen enter three piecewise functions.



At $y1$ press **ENTER**. Find **when**(in the **CATALOG** quickly by pressing **CATALOG** [.]). This shows the notation: **when(condition, true, false)**

For $y1$, type **when**($x < 1, 1, a$)| $a = 5$

The “such that” bar key ($[|]$) is to the left of the $[7]$ key.

$$y1(x) = \begin{cases} 1, & x < 1 \\ a, & x \geq 1 \end{cases} | a = 5$$

For $y2$, type **when**($x < 1, x + 2, a * x^2$)| $a = 5$

$$y2(x) = \begin{cases} x + 2, & x < 1 \\ a \cdot x^2, & x \geq 1 \end{cases} | a = 5$$

For $y3$, type **when** ($x < 2, 2 \sin((x - 1)\frac{\pi}{2}), a + 3 \sin((x - 4)\frac{\pi}{2})$)| $a = 5$

$$y3(x) = \begin{cases} 2 \sin((x - 1)\frac{\pi}{2}), & x < 2 \\ a + 3 \sin((x - 4)\frac{\pi}{2}), & x \geq 12 \end{cases} | a = 5$$

6. Graph one function at a time by using $F4$ to have only one function checked at a time.

On a graph screen examine both sides of where the discontinuity exists using $F3$ Trace.

7. For Problems 1 and 2 below, use $[\blacklozenge]F4$ to have table settings of $\text{tblStart} = 0.98$ and $\Delta\text{tbl} = 0.01$, to numerically examine the left and right-hand limits. Be sure to press **ENTER** to save changes before pressing $[\blacklozenge]F5$ to view the table.

For Problems 1, 2, and 3 estimate the limits graphically and numerically using trace and table.

Problem 1

$$y1(x) = \begin{cases} 1, & x < 1 \\ a, & x \geq 1 \end{cases} \quad | a = 5$$

$$\lim_{x \rightarrow 1^-} y1(x) \approx \underline{\hspace{2cm}}$$

$$\lim_{x \rightarrow 1^+} y1(x) \approx \underline{\hspace{2cm}}$$

Try other values for a in the graph of $y1(x)$ to find what a makes $\lim_{x \rightarrow 1} y1(x)$ exist. On the $Y =$ screen, press **ENTER** when $y1$ is highlighted. Press \blacktriangleright and then backspace \blacktriangleleft to try different values for a . Graph it to see if appear continuous.

$$a = \underline{\hspace{2cm}}$$

Problem 2

$$y2(x) = \begin{cases} x + 2, & x < 1 \\ a \cdot x^2, & x \geq 1 \end{cases} \quad | a = 5$$

$$\lim_{x \rightarrow 1^-} y2(x) \approx \underline{\hspace{2cm}}$$

$$\lim_{x \rightarrow 1^+} y2(x) \approx \underline{\hspace{2cm}}$$

Try other values for a in the graph of $y2(x)$ to find what a makes $\lim_{x \rightarrow 1} y2(x)$ exist.

$$a = \underline{\hspace{2cm}}$$

Show calculations of the left hand limit and the right hand limit to verify that your value for a makes the limit exist.

Problem 3

$$y3(x) = \begin{cases} 2 \sin\left((x-1)\frac{\pi}{2}\right), & x < 2 \\ a + 3 \sin\left((x-4)\frac{\pi}{2}\right), & x \geq 2 \end{cases} \quad | a = 5$$

$$\lim_{x \rightarrow 2^-} y3(x) \approx \underline{\hspace{2cm}}$$

$$\lim_{x \rightarrow 2^+} y3(x) \approx \underline{\hspace{2cm}}$$

Try other values for a in the graph of $y3(x)$ to find what a makes $\lim_{x \rightarrow 2} y3(x)$ exist.

$$a = \underline{\hspace{2cm}}$$

Show calculations of the left hand limit and the right hand limit to verify that your value for a makes the limit exist.

Extension –Continuity

A function is continuous at $x = c$ if:

- $f(c)$ exists
- $\lim_{x \rightarrow c} f(x)$ exists, and
- $\lim_{x \rightarrow c} f(x) = f(c)$

Use CAS to algebraically solve for a that makes

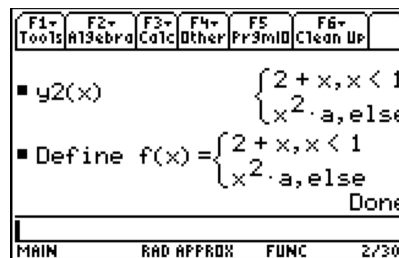
(a) $\lim_{x \rightarrow 1} y_2(x)$ exist

(b) $\lim_{x \rightarrow 2} y_3(x)$ exist

Then prove each function is continuous.

Key press help:

- Begin by pressing **HOME**. Clean Up the screen by pressing $2^{nd}[F1]$. Choose **NewProb** and press **ENTER** to put this on the command line and **ENTER** to execute the command.
- Type $y_2(x)$ **ENTER**. The Define command is under the $F4$ menu. Type **Define** $f(x) =$, then up arrow to highlight the output from the previous line. Press **ENTER** on the highlighted piecewise function to copy it down to the command line.



- To solve a right sided limit, press $F3 >$ **limit** (. On the command line enter **limit**($f(x), x, 1, 1$) **ENTER**.
- Now, press $F2$ **ENTER** to select **solve**(. Then up arrow to select the input from the previous line, press **ENTER**. Next type [-]. Up arrow to the input again and press **ENTER**. This time put a negative (-) in front of the last 1. Finally type [,] **ALPHA** [-] and close the parentheses. This method will enable you to quickly enter **solve**(**limit**($f(x), x, 1, 1$) = **limit** ($f(x), x, 1, -1$), a).

CHAPTER

3**SE Differentiation - TI****Chapter Outline**

- 3.1 GETTING' THE SWING**
 - 3.2 MOVE THOSE CHAINS**
 - 3.3 IMPLICIT DIFFERENTIATION**
-

The activities below are intended to supplement our CK-12 Calculus FlexBook® textbook, [Chapter 2](#) .

3.1 Getting' the Swing

This activity is intended to supplement Calculus, Chapter 2, Lesson 4.

Part 1 –Warm-up

In y1, enter $\cos(x)$. Press $F3$ and select **7: ZoomTrig**. Use the graph to answer the following questions.

1. What is the range?
2. What is the amplitude? $A =$
3. What is the period? $T =$

Now change your calculator mode to split screen. Press 3 and select **TOP-BOTTOM** for **Split Screen**. For **Split 1 App**, select $Y =$ **Editor**. For **Split 2 App**, select **Graph**. In y2, enter an equation in the form $y = A \cdot \cos(B \cdot x) + C$, where A , B , and C are integers. Press $2 + \alpha$ to swap applications to see the graph screen update. Press $2 + \alpha$ again to go back to the $Y =$ **Editor** to modify your equation. To answer the following questions, modify the corresponding variable to observe the changes each variable has to the equation.

4. Describe the effect of increasing A .
5. Describe the effect of increasing C .
6. Describe the effect of increasing B .
7. What is the relationship between B and the period, T ?
8. If a positive D shifts the graph to the right D units, what is the general sinusoidal equation for which this is true?

Part 2 –Collect & Analyze Data

You will collect data of a pendulum swinging. Using the skills reviewed in the warm-up, write a cosine function that models the data collected. Estimate the amplitude and period and phase shift, D , to the nearest tenth. If a motion detector is not available, use the lists *time*, *distance*, and *velocity* from your teacher and graph a function to model that data. To collect data, complete the following steps:

- Using an I/O cable, connect the motion detector to the graphing calculator.
- On the **HOME** screen, run the *Ranger* program. Select **1:Setup/Sample....** Use the settings that appear to the right and press **ENTER**.
- Position the motion detector so that it is facing the pendulum, swing the pendulum, and press **ENTER** to begin collecting data.



- If your data doesn't look sinusoidal, press **ENTER** and select **3:Repeat Sample** to repeat the trial. Then press **ENTER** to begin collecting data again.
- Model the distance-time data with a function. Derive the velocity and acceleration equations. Select **7:Quit** when you are finished.

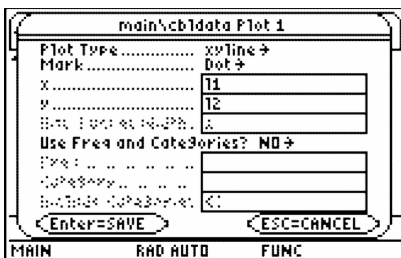
Record your position, velocity and acceleration equations for your experiment data here:

$y =$

$v =$

$a =$

Confirm your position and velocity equations by graphing them. To confirm your position equation, enter your equation in $y1$ and select to show Plot 1 as shown to the right. To plot the velocity-time graph, use $L1$ for time and $L3$ for velocity. For the acceleration-time graph, use $L1$ for time and $L4$ for acceleration.



3.2 Move those Chains

This activity is intended to supplement Calculus, Chapter 2, Lesson 5.

Problem 1 –Derivative Using the Power Rule

Recall the Power Rule $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$.

1. Based on the Power Rule, what do you think the derivative of $f(x) = (2x + 1)^2$ is?

Graph the derivative of the function and your conjecture about the derivative. Go to the $Y =$ Editor. In $y1$, type $(2x + 1)^2$. In $y2$, type **nDeriv**($y1(x), x$). To access the **nDeriv** command, go to the Math menu (2^{nd} [**MATH**]) and select **B:Calculus > A:nDeriv**(. In $y3$, type your conjecture for the derivative of $f(x) = (2x + 1)^2$. Highlight $y1$ and press $F4$ to unselect this function, and press $\blacklozenge F3$ to graph $y2$ and $y3$. Note: The graph may take a minute to appear. If your conjecture is correct, the graphs of $y2$ and $y3$ will coincide. If your conjecture is incorrect, the graphs of $y2$ and $y3$ will not coincide.

2. Was your conjecture correct? If not, how can you change your conjecture to make it correct?

3. Expand the binomial $(2x + 1)^2$. Take the derivative of each term. How does this compare with your answer to Question 1?

Problem 2 –The Chain Rule

The following are 'true' statements that can be verified on the TI-89.

$$d((5x + 7)^3, x) = 3 \cdot (5x + 7)^2 \cdot 5x \text{ true}$$

$$d((x^3 + 7)^5, x) = 5 \cdot (x^3 + 7)^4 \cdot 3x^2 \text{ true}$$

$$d((x^2 + 6)^4, x) = 4 \cdot (x^2 + 6)^3 \cdot 2x \text{ true}$$

4. What patterns do you see? Using any information that you can infer from these statements, create a rule for finding the derivative of these functions. Discuss the patterns you see and the rule you created with a partner.

5. Using your rule from Question 4, what is $\frac{d}{dx}((3x + 2)^2)$?

Verify your answer by typing your statement on the entry line of your TI-89. If you are correct, the TI-89 will return the word, 'true'. If you are incorrect, the TI-89 will return a false statement. If you are incorrect, try again by editing your statement. You can copy your last command by selecting 2^{nd} **ENTER**.

6. What is $\frac{d}{dx}((7x + 2)^3)$? Verify your answer.

7. What is $\frac{d}{dx}((4x^2 + 2x + 3)^4)$? Verify your answer.

The derivative rule you have just observed is called the **Chain Rule**. It is used to take the derivative of composite functions. The Chain Rule is $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$. First, take the derivative of the "outside function" at $g(x)$. Then, multiply this by the derivative of the "inside function."

8. Use the Chain Rule to create three additional true statements. Verify your answers.

Problem 3 –Homework Problems

Evaluate the following derivatives using the Chain Rule. Verify your answers.

1. $\frac{d}{dx}((4x^3 + 1)^2) =$

2. $\frac{d}{dx}((-5x + 10)^7) =$

3. $\frac{d}{dt}((2t^5 - 4t^3 + 2t - 1)^2) =$

4. $\frac{d}{dx}((x^2 + 5)^{-2}) =$

5. $\frac{d}{dz}((z^3 - 3z^2 + 4)^{-3}) =$

3.3 Implicit Differentiation

This activity is intended to supplement *Calculus, Chapter 2, Lesson 6*.

Problem 1 –Finding the Derivative of

The relation, $x^2 + y^2 = 36$, in its current form *implicitly* defines two functions, $f_1(x) = y$ and $f_2(x) = y$. Find these two functions by solving $x^2 + y^2 = 36$ for y .

$$f_1(x) =$$

$$f_2(x) =$$

Substitute the above functions in the original relation and then simplify.

$$x^2 + (f_1(x))^2 = 36$$

$$x^2 + (f_2(x))^2 = 36$$

This confirms that $f_1(x)$ and $f_2(x)$ *explicitly* defines the relation $x^2 + y^2 = 36$.

Graph $f_1(x)$ and $f_2(x)$ on the same set of axis and then draw it in the space to the right. Imagine that you were asked to find the slope of the curve at $x = 2$.

- Why might this question be potentially difficult to answer?
- What strategies or methods could you use to answer this question?

One way to find the slope of a tangent drawn to the circle at any point (x, y) located on the curve is by taking the derivative of $f_1(x)$ and $f_2(x)$.

$$\frac{dy}{dx}f_1(x) =$$

$$\frac{dy}{dx}f_2(x) =$$

Check that your derivatives are correct by using the **Derivative** command (press **F3:Calc >1:d(differentiate)** on the *Calculator* screen.

Substitute 2 for x to determine the slope of the tangents to $x^2 + y^2 = 36$ at $x = 2$.

$$\frac{dy}{dx}f_1(2) =$$

$$\frac{dy}{dx}f_2(2) =$$

Another way to find the slope of a tangent is by finding the derivative of $x^2 + y^2 = 36$ using *implicit differentiation*. On the *Calculator* screen press **F3:Calc >D:impDif(** to access the **impDif** command. Enter **impDif** ($x^2 + y^2 = 36, x, y$) to find the derivative.

$$\frac{dy}{dx} =$$

Use this result to find the slope of the tangents to $x^2 + y^2 = 36$ at $x = 2$. First you will need to find the y -values when $x = 2$.

$$\frac{dy}{dx}(2,y) = \qquad \qquad \qquad \frac{dy}{dx}(2,y) =$$

- Is your answer consistent with what was found earlier?
- Rewrite the implicit differentiation derivative in terms of x . Show that, for all values of x and y , the derivatives of $f_1(x)$ and $f_2(x)$ that you found earlier are equal to the result found using the **impDif** command.

Problem 2 –Finding the Derivative of

To find the derivative of a relation $F(x,y)$, take the derivative of y with respect to x of each side of the relation. Looking at the original example, $x^2 + y^2 = 36$, we get:

$$\begin{aligned} \frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(36) \\ \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= \frac{d}{dx}(36) \end{aligned}$$

Evaluate the following and by hand.

$$\frac{d}{dx}(x^2) = \qquad \qquad \qquad \frac{d}{dx}(36) =$$

Use the **Derivative** command to find $\frac{d}{dx}(y^2)$. Set up the expression up as $\frac{d}{dx}(y(x)^2)$. Notice that $y(x)$ is used rather than just y . This is very important because it reminds the calculator that y is a function of x .

$$\frac{d}{dx}(y^2) =$$

You have now evaluated $\frac{d}{dx}(x^2)$, $\frac{d}{dx}(y^2)$, and $\frac{d}{dx}(36)$. Replace these expressions in the equation $\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(36)$ and solve for $\frac{dy}{dx}$.

Compare your result to the one obtained using the **impDif** command.

Problem 3 –Finding the Derivative of

The relation, $y^2 + xy = 2$, can also be solved as two functions, $f_1(x)$ and $f_2(x)$, which *explicitly* define it.

- What strategy can be used to solve $y^2 + xy = 2$ for y ?

Solve $y^2 + xy = 2$ for y and use the **Solve** command (press **F2:Algebra >1:solve()**) to check your answer.

The derivative of $y^2 + xy = 2$ can then be found by taking the derivatives of $f_1(x)$ and $f_2(x)$. However, the derivative can be found more easily using implicit differentiation.

Use implicit differentiation to find the derivative of $y^2 + xy = 2$. Check your result by using the **impDif** command. (*Hint:* The product rule must be used to find the derivative of xy .)

$$\frac{dy}{dx} =$$

Use the derivative you found for $y^2 + xy = 2$ to calculate the slope at $x = -6$. First you will need to find the y -values when $x = -6$.

$$\frac{dy}{dx}(-6, y) =$$

$$\frac{dy}{dx}(-6, y) =$$

Verify your result graphically. Graph the two functions, $f_1(x)$ and $f_2(x)$. Then use the slopes and points to graph each tangent line.

Extension –Finding the Derivative of

The relation $x^3 + y^3 = 6xy$ cannot be solved explicitly for y . In this case implicit differentiation must be used.

- Find the derivative of $x^3 + y^3 = 6xy$ and use the **impDif** command to verify your result.

$$\frac{dy}{dx} =$$

Use this result to find the slope of the tangents to $x^3 + y^3 = 6xy$ at $x = 1$. (*Hint:* Use the **solve** command to find the y values that correspond to $x = 1$.)

CHAPTER 4**SE Applications of Derivatives - TI****Chapter Outline**

- 4.1 XTREME CALCULUS**
 - 4.2 HELICOPTER BUNGEE JUMP**
 - 4.3 OPTIMIZATION**
 - 4.4 LINEAR APPROXIMATIONS**
-

The activities below are intended to supplement our CK-12 Calculus FlexBook® textbook, [Chapter 3](#) .

4.1 Xtreme Calculus

This activity is intended to supplement Calculus, Chapter 3, Lesson 2.

Max & Min

In this activity, you will explore relative maximums and minimums by drawing a tangent line to a curve and making observations about the slope of the tangent line.

Press **APPS**, select the *Text Editor* application, and open *xtreme1*. Press **F4** to execute each command line and read the questions or instructions. (*xtreme1.89t* and *tanimat2.89p* must be installed on your TI-89.)

When the *tanimate2* program begins, choose **Interactive**. Press **ALPHA** 10 to find the slope 10 times. (You can choose a smaller number next time.) Select the **Tangent Only** option. Arrow to a point where you want to find the slope and press **ENTER**, or type a value and press **ENTER** twice. Observe the slope of the tangent line and determine the critical number(s) of the function.

1. What is (are) the critical number(s) of $y_1(x)$?
2. What occurs at each of the critical numbers of $y_1(x)$?

From the Main Menu arrow to the right, select **QUIT** and press **ENTER**. You will return to the script. Continue pressing **F4** to progress down and set up the next function.

For the graph that has a cusp:

3. What is (are) the critical number(s) of $y_1(x)$?
4. What occurs at each of the critical numbers of $y_1(x)$?

For the cubic function:

5. What is (are) the critical number(s) of $y_1(x)$?
6. What occurs at the critical numbers of $y_1(x)$?
7. Does a relative extreme value occur at every critical number? Predict a way to help determine whether or not there is a relative extreme value at a critical number.

Think about how you can tell if a critical number will be at a relative maximum, a relative minimum, or neither. When using *tanimat2* for the quadratic opening down, move the point of tangency along the curve. Find the slope of the tangent at many locations to observe what happens.

8. When the point of tangency is to the left side of the relative maximum, will the slope of the tangent line be positive, negative, or zero?
9. What about when the point of tangency is to the right of the relative maximum?

Quit *tanimat2* and return to the *xtreme1* script. Repeat the process for the quadratic function opening up.

10. For this function, when the point of tangency is to the left side of the relative minimum, will the slope of the tangent line be positive, negative, or zero?
11. What about when the point of tangency is to the right of the relative minimum?

Fill in the blanks of the following sentences.

12. Let f have a critical number at $x = c$. If $f'(x) > 0$ on an open interval extending left from c , and $f'(x) < 0$ on an open interval extending from right of c , then f has a _____ at $x = c$.
13. Let f have a critical number at $x = c$. If $f'(x) < 0$ on an open interval extending left from c , and $f'(x) > 0$ on an open interval extending from right of c , then f has a _____ at $x = c$.
14. Let f have a critical number at $x = c$. If $f'(x)$ has the same sign on an open interval extending left from c and on an open interval extending right from c , then f has a _____ at $x = c$.

Extension

15. How many relative extrema can an n th degree polynomial have? Explain.

4.2 Helicopter Bungee Jump

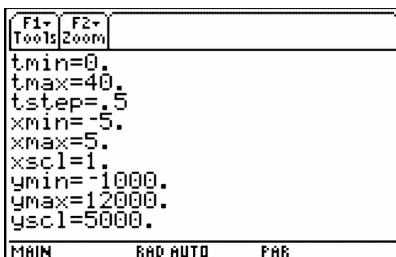
This activity is intended to supplement *Calculus, Chapter 3, Lesson 4*.

Part 1 – Bungee Jump

In June 2001, the record for the longest bungee jump was shattered when a stuntman jumped from a helicopter hovering over 10,000 feet. This “Mile Long Bungee Jump” is illustrated using the following parametric equations:

$$x1(t) = 1$$

$$y1(t) = -1200e^{-0.1t+1.5} \cos(0.2(t-18)) + 5200.$$



Enter these equations in your calculator, then change the style for this set of parametric equations to **F6: Style >6:Path**. Set your window settings to match those to the right. Press \blacklozenge **GRAPH** to watch the simulation.

For positive time t , the position of the bungee jumper can be modeled by the following function:

$$y(t) = -1200e^{-0.1t+1.5} \cos(0.2(t-18)) + 5200, \text{ when}$$

$0 < t < 40$. Enter this function in $y1(x)$, using x in place of t . Take the derivative of the function twice to find the velocity and acceleration functions.

1. Enter the following command on the **HOME** screen: $\text{solve}(d(y1(t),t) = 0, t) | 0 < t < 40$. What is the significance of this result? Notice the argument “ t ” is needed twice and the “such that” symbol (“|”) limits the domain.
2. What physical quantity is given by the second derivative of position?
3. Within the first 40 seconds, when do (does) the extrema for the velocity occur? Show your work.
4. The third derivative of position with respect to time is known as *jerk*. After the first time the velocity is zero, when does jerk have the largest magnitude?
5. When is the downward velocity at a maximum? What is the speed at that time?

Enter the velocity function in $y2$, the acceleration function in $y3$, and the jerk function in $y4$. Examine the position-time graph, the velocity-time graph, and the acceleration-time graph. Adjust the graphing window as necessary.

6. Write at least two complete sentences relating position-time, velocity-time, and acceleration-time graphs from the helicopter bungee jump situation.

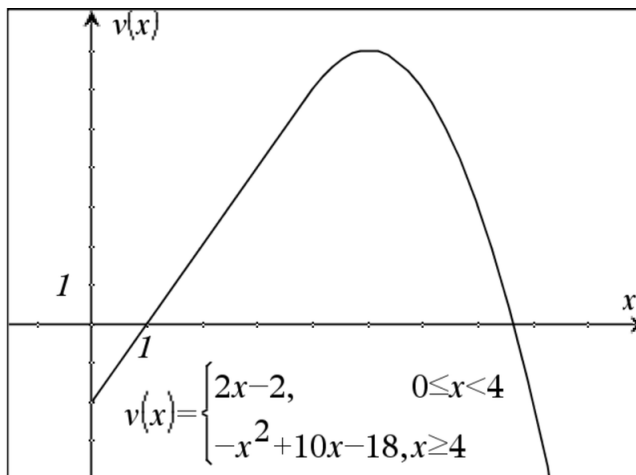
On the acceleration-time graph, the mathematical model is not realistic for the first 4 seconds, but it is after that. Change the window settings so that you can no longer see the first 4 seconds of the acceleration-time graph.

7. After 4 seconds, what is the maximum number of g's. Use the graph to justify your answer. Remember that $1 g = 32 ft/s^2$.
8. What is the point of inflection where the graph changes from concave up to concave down in the first 40 seconds? Use the **Inflection Point** tool (**F5:Math >8:Inflection**).

Part 2 – Graphically examine another situation

Let s be the function $s(t) = \int_0^t v(x) dx$.

9. $s(1) =$
10. $s'(1) =$
11. $s''(1) =$



12. Use calculus to find when v is a maximum. Show your work.
13. For $0 < x < 7$, when is the graph of s concave up? Explain your reasoning.
14. For $0 < x < 7$, when is the graph of s decreasing? Explain your reasoning.

4.3 Optimization

This activity is intended to supplement Calculus, Chapter 3, Lesson 7.

Problem 1 –Optimization of distance and area

Graph the line $y = 4x + 7$. Find the point on the line that is closest to the origin.

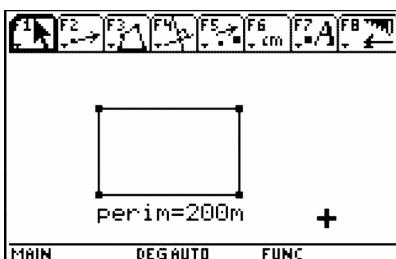
- What point do you think minimizes the distance from the point to the origin?
- What function are you trying to minimize?
- What is the constraint?
- Write the function to minimize using one variable.

On the Home screen, find the exact coordinates that minimize the distance using the **Derivative** and **Solve** commands. To do this, find the first derivative, solve to find the critical value(s), and then find the second derivative to confirm a minimum.

- What are the x - and y -coordinates of the point?
- What is the minimum distance?

Find the dimensions of a rectangle with perimeter 200 meters whose area is as large as possible.

- What dimensions do you think maximize the area?
- What function are you trying to maximize?



- What is the constraint?
- Write the function to maximize using one variable.

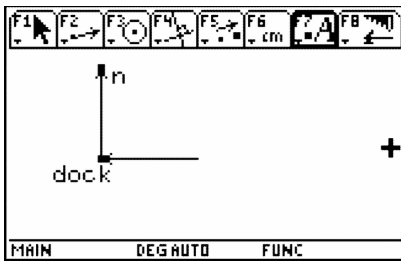
Find the dimensions that maximize the area using the **Derivative** and **Solve** commands.

- What are the dimensions of the rectangle?

Problem 2 –Optimization of time derivative problems

A boat leaves a dock at 1 pm and travels north at a speed of 20 km/h. Another boat has been heading west at 15 km/h. It reaches the same dock at 2 pm. At what time were the boats closest together? Use t for time.

- What is the position function for the boat heading north? West?



- What function are you trying to minimize?
- What is the constraint?
- Write the function to minimize using one variable.

Find the time at which the distance between the two boats is minimized using the **Derivative** and **Solve** commands.

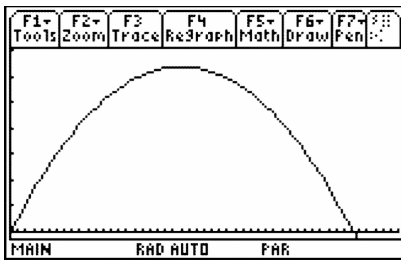
- What is the minimum distance?
- What is the time at which this occurs? Remember to convert the value of t to minutes.

Extension –Parametric function

A projectile is fired with the following parametric functions:

$$x = 500 \cos(30^\circ)t, y = 500 \sin(30^\circ)t - 4.9t^2$$

- What is the time when the projectile hits the ground?
- How far does it travel horizontally?
- What is the maximum height that it achieves?



4.4 Linear Approximations

This activity is intended to supplement *Calculus, Chapter 3, Lesson 8*.

Part 1 –Introduction

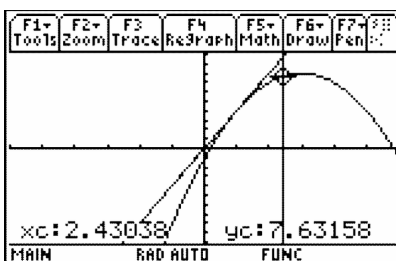
Linear approximation uses a tangent line to estimate values of a function near the point of tangency. For this reason, *linear approximation* is also referred to as *tangent line approximation*.

On the graph to the right, let a be the point where the tangent touches the graph, $L(x)$ be the tangent, and $f(x)$ be the function.

On the picture, the point x is the x - coordinate of the vertical line.

Draw a vertical line from a to the x -axis.

Draw horizontal lines from a , $f(x)$, and the intersection of the vertical line with the tangent line.



At this stage, you should have three points on the y -axis: $f(a)$, $f(x)$, $L(x)$. Label them.

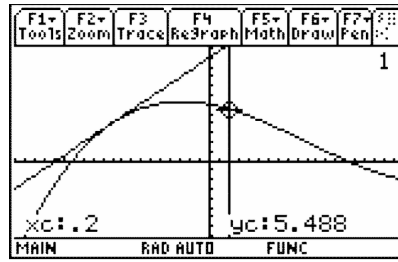
- Which of these points can you use to represent the estimate, or linear approximation, of $f(x)$ near a ?
- How can you use these labels to represent the *error* associated with this estimate?
- Is this estimate an *overestimate* or an *underestimate*? Explain.

Part 2 –Investigating linear approximation

On the graph at the right, $f(x) = x^3 - 3x^2 - 2x + 6$ is shown. The tangent line at $a = -1$ is $L(x) = 7x + 11$. The trace object q is at the point $(0.2, 5.488)$.

If you draw a vertical line from the x -axis through this point, you will get the point $p = (0.212658, 12.4886)$.

The distance $pq = 7.039997$ or 7.04 .



- What numerical value represents the linear approximation of $f1(q)$ near $a = -1$?
- What numerical value gives the error associated with this linear approximation?
- What is the true value of $f1(q)$?
- Is this an underestimation or an overestimation?

Repeat the process above and complete the table below for the x -values given.

TABLE 4.1:

p	distance pq	linear approx. of $f1(q)$	real value of $f1(q)$	error	underestimation/overestimation
$x = -0.2$					
$x = -0.5$					
$x = -0.6$					
$x = -1.2$					

- At what x -value(s) is the error less than 0.5?
- What do you notice about graph of the function and the graph of the tangent line as you get close to the point of tangency?
- Based on your observations, explain why the relationship between a tangent and a graph at the point of tangency is often referred to as *local linearization*.

Typically, you will have a function but not a graph to find the linear approximation.

- Find the derivative of $f1(x) = x^3 - 3x^2 - 2x + 6$. Evaluate it at $x = -1$. This is the slope of the line. Use the slope and the point $(-1, 4)$ to get the equation of the line.
- The tangent line $L(x) =$
- What is $L(-1.03)$? What does this value represent?
- Calculate the error with this estimate.

Part 3 –Underestimates versus overestimates

Graph the function $f1(x) = x^3 - 3x^2 - 2x + 6$ and place a tangent line $a = 1$.

- If you were to draw a point p on the graph to the left of $a = 1$, is the approximation an *overestimate* or an *underestimate*?
- If you draw a point p on the graph to the right of $a = 1$, is the approximation an *overestimate* or an *underestimate*?
- What is the significance of the point of tangency?

- Generalize your findings about when a linear approximation produces an overestimate and when it produces an underestimate.

Part 4 –Finding intervals of accuracy

How close to -1 must x be for the linear approximation of $f(x) = x^3 - 3x^2 - 2x + 6$ at $a = -1$ to be within 0.2 units of the true value of $f(x)$?

Graph $f_2 = f + 0.2$ and $f_3 = f - 0.2$ with f and the tangent line.

- How would you use the graphs to answer the question posed in this problem?
- How close to -1 must x be for the linear approximation of $f(x) = x^3 - 3x^2 - 2x + 6$ at $a = -1$ to be within 0.2 units of the *true* value of $f(x)$?

Now we want to ask the same questions when the point of tangency is at $a = 1$.

- How does this situation differ from the one we just had?
- Use graphical or algebraic methods to find an interval that ensures the linear approximation at $a = 1$ is accurate to within 0.2 units of $f(x)$.

CHAPTER

5**SE Integration - TI****Chapter Outline**

5.1 **EXPLORING AREA UNDER THE CURVE**

5.2 **SUM RECTANGLES**

5.3 **FTC CHANGED HISTORY**

The activities below are intended to supplement our CK-12 Calculus FlexBook® textbook, [Chapter 4](#) .

5.1 Exploring Area under the Curve

This activity is intended to supplement *Calculus, Chapter 4, Lesson 3*.

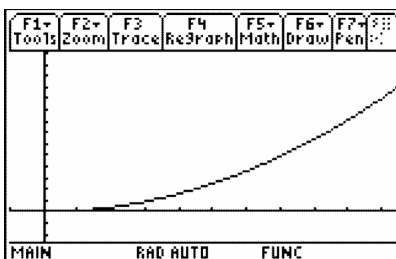
Problem 1 –Explore and discover

Graph the curve $y = x^2$.

Your challenge is to think of *at least* two ways to estimate the area bounded by the curve $y = x^2$ and the x -axis on the interval $[0, 1]$ using rectangles. Use the following guidelines:

- all rectangles must have the same width
- you must build all your rectangles using the same methods
- the base of each rectangle must lie on the x -axis

Graph $y = x^2$ and set your window to $[-0.1, 1]$ for x and $[-0.2, 1.3]$ for y . Draw your first and second method on the graphs below. For each method calculate the following:



- Number of rectangles
- Height and width of each one
- Area of each
- Sum of the area
- Which method did a better job?
- How could you improve on it?

In the following problem, you will examine three common techniques that use rectangles to find the approximate area under a curve. Perhaps you discovered some of these techniques during your exploration in the above problem. The first problem uses rectangles whose right-endpoints lie on the curve $y = x^2$.

Problem 2 –Using five right-endpoint rectangles

Divide the interval $[0, 1]$ into five equal pieces. Enter the information for each interval or rectangle in the table below. Remember that the right endpoint is the x -value and the height is the y -value of the right endpoint on the curve.

TABLE 5.1:

Interval	Right Endpoint	Height	Area
----------	----------------	--------	------

- Calculate this sum.
- Now add up the numbers in the Area column.

The formula that can be used to express the total area is:

$$R_5 = 0.2 \cdot f1(0.2) + 0.2 \cdot f1(0.4) + 0.2 \cdot f1(0.6) + 0.2 \cdot f1(0.8) + 0.2 \cdot f1(1.0)$$

or

$$R_5 = 0.2[f1(0.2) + f1(0.4) + f1(0.6) + f1(0.8) + f1(1.0)]$$

- Are these two numbers the same or different?

Another way to find the area of the rectangles is using sigma notation.

- Write the notation in the $\sum_{x=1}^5 x^2$ form. Adjust what is being summed.

To sum it on the calculator, use **Home >F3:Calc >4:Sigma** for the command with the format:

Σ (expression, variable, lower limit, upper limit)

- Does this agree with the answer for the area you found previously?

Problem 3 –Using five left-endpoint rectangles

This problem uses rectangles whose left-endpoints lie on the curve $y = x^2$.

Divide the interval $[0, 1]$ into five equal pieces. Enter the information for each interval in the table below. Remember that the left endpoint is the x -value and the height is the y -value of the left endpoint on the curve.

TABLE 5.2:

Interval	Left Endpoint	Height	Area
----------	---------------	--------	------

- Calculate this sum.
- Now add up the numbers in the Area column.

The formula that can be used to express this area is:

$$L_5 = 0.2 \cdot f1(0) + 0.2 \cdot f1(0.2) + 0.2 \cdot f1(0.4) + 0.2 \cdot f1(0.6) + 0.2 \cdot f1(0.8)$$

or

$$L_5 = 0.2[f1(0) + f1(0.2) + f1(0.4) + f1(0.6) + f1(0.8)]$$

- Are these two numbers the same or different?

- What is the sigma notation for the area of the rectangles?
- Use the calculator to find the sum. Does this result agree with the answer above?

Problem 4 –Using five midpoint rectangles

We will now investigate a *midpoint* approximation. How would you draw five rectangles, with equal width, such that their midpoints lie on the curve $y = x^2$?

Divide the interval $[0, 1]$ into five equal pieces. Enter the information for each interval or rectangle in the table below. Remember that the midpoint is the x -value and the height is the y -value of the midpoint on the curve.

TABLE 5.3:

Interval	Midpoint	Height	Area
----------	----------	--------	------

- Calculate this sum.
- Now add up the numbers in the Area column.
- Are these two numbers the same or different?

The formula that can be used to express this area is:

$$M_5 = 0.2 \cdot f(0.1) + 0.2 \cdot f(0.3) + 0.2 \cdot f(0.5) + 0.2 \cdot f(0.7) + 0.2 \cdot f(0.9)$$

or

$$M_5 = 0.2[f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)]$$

- What is the sigma notation for the area of the rectangles?
- Use the calculator to find the sum. Does this result agree with the answer above?

Problem 5- Summarize your findings

In this activity, you explored three different methods for approximating the area under a curve. The exact area under the curve $y = x^2$ on the interval $[0, 1]$ is $\frac{1}{3}$ or 0.333.

- Which approximation produced the best estimate for the actual area under the curve?
- Describe which factors contribute to left, right, and midpoint rectangles giving *overestimates* versus *underestimates*.
- What can you do to ensure that all three of these techniques produce an answer that is very close to $\frac{1}{3}$? Test your conjecture by using evaluating a sum that produces a much more accurate answer.

5.2 Sum Rectangles

This activity is intended to supplement Calculus, Chapter 4, Lesson 4.

Part 1 –Graphical Riemann Sums

On the **HOME** screen, type **area()** to start the Area Approximation program.

To use the program, select to view the approximation either graphically or numerically. Next, enter the equation that you are examining and press **ENTER**. You will be prompted for the minimum value, maximum value, and the number of steps. Then, select the approximation method and press **ENTER**. The approximation will be displayed on the screen. You can then select to enter another function, do another approximation or to exit the program.

Use the Area Approximation program to graphically examine functions to complete this part of the activity.

- For $y_1(x) = -0.5x^2 + 40$, how do the left, midpoint, and right Riemann sums compare? Explain why.
- Describe what happens to the left, midpoint, and right Riemann sums as you increase the number of subintervals, n .
- Is the midpoint Riemann sum an over or under approximation if the graph is:
 - increasing and concave down? ____ over ____ under
 - increasing and concave up? ____ over ____ under
 - decreasing and concave down? ____ over ____ under
 - decreasing and concave up? ____ over ____ under

After graphically exploring (especially with a small number of subintervals), explain why.

Part 2 –Summation notation

Examine the function $y_1(x) = -0.5x^2 + 40$.

- The thickness of each rectangle is $\Delta x = h = \frac{b-a}{n}$. If $a = 1$, $b = 6$, and $n = 5$. What is Δx ?
- Expand $\sum_{i=1}^5 (1 \cdot y(a + (i-1) \cdot 1))$ by writing the sum of the five terms and substitute $i = 1, 2, 3, 4$, and 5 .
- Explain why this is the summation notation for **LEFT** Riemann sums and not the **RIGHT**.
- Let $y(x) = -0.5x^2 + 40$, $a = 1$, and $b = 6$. Write the sigma notation and use the **HOME** screen to evaluate the left Riemann sum for 10, 20, 50, and 100 subintervals.
 - $n = 10$
 - $n = 20$
 - $n = 50$
 - $n = 100$

Extension –Area Programs

Use the Area Approximation program to answer the following questions.

1. Let $y(x) = x^2$, $a = 1$, and $b = 6$. Write the results for midpoint and trapezoid area approximations when:
 - a. $n = 10$
 - b. $n = 100$
 - c. $n = 500$
2. Compare the above midpoint and trapezoid values with the actual area.

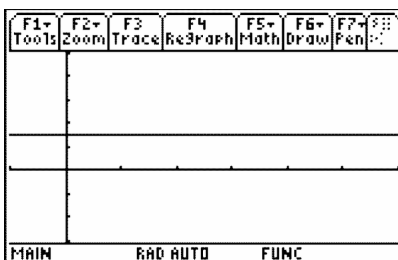
5.3 FTC Changed History

This activity is intended to supplement Calculus, Chapter 4, Lesson 6.

Problem 1 –Constant integrand

Suppose you have the function $y = 1.5$ as seen at the right. How will the area under the curve change as we go from 0 to x ? Find the area of the by evaluating the definite integral $\int_0^x 1.5 dt$.

For each value of x , you are looking at a rectangle with x for the length and 1.5 for the height.



- Use the **Integrate** command (**HOME** > **F3:Calc** > **2:Integrate**) to complete the table.

TABLE 5.4:

x	$\int_0^x 1.5 dt$
1	
2	
3	
4	
5	

- If $x = 0$, what is $\int_0^x 1.5 dt$? Why?

- For every 1 unit that x changes, how much does $\int_0^x 1.5 dt$ change?

- If you were to graph the ordered pairs $\left(x, \int_0^x 1.5 dt\right)$, what would the graph look like?

Use the **Stats/List Editor** to enter the data in the table above into *list1* and *list2*. Then plot the data.

- What does your graph look like? Was this graph what you predicted in Question 4?
- If you changed the integrand from 1.5 to 0.5, what would the graph of $\left(x, \int_0^x 0.5 dt\right)$ look like?

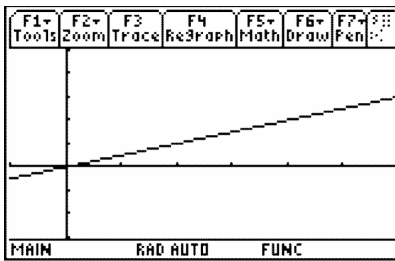
Problem 2 –Non-Constant Integrand

Suppose you have the function $y = \frac{x}{2}$ as seen below. How will the area under the curve change as you go from 0 to x ? Find the area of the triangle by hand or by evaluating the definite integral $\int_0^x \frac{t}{2} dt$.

7. Complete the table.

TABLE 5.5:

x	$\int_0^x \frac{t}{2} dt$
1	
2	
3	
4	
5	



8. If $x = 0$, what is $\int_0^x \frac{t}{2} dt$? Why?

9. Explain why, when x increases by 1, the value of $\int_0^x \frac{t}{2} dt$ does not increase by the same amount every time?

10. Is the graph of $\left(x, \int_0^x \frac{t}{2} dt\right)$ linear? Explain.

Problem 3 –An Integrand That Changes Sign

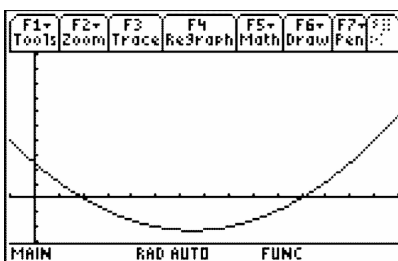
In the previous exercises, the function was positive over the interval. This time you are going to examine a function which changes sign, $y = \frac{x^2 - 13x + 22}{9}$. How will the area under the curve change as we go from 0 to x ? Find the area of the by evaluating the definite integral $\int_0^x \frac{t^2 - 13t + 22}{9} dt$. Complete the table.

TABLE 5.6:

x	$\int_0^x \frac{t^2 - 13t + 22}{9} dt$
1	
2	
3	
4	

TABLE 5.6: (continued)

x	$\int_0^x \frac{t^2 - 13t + 22}{9} dt$
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	



11. At what value of x does the integral's value begin to decrease?
12. a. What are all the values of x for which the definite integral's value is decreasing?
b. What is true at these values of x ?
13. a. What are all the values of x for which the integral's value is increasing?
b. What is true of the integrand at these values of x ?
14. a. What is the smallest value of the integral, and at what value of x is this reached?
b. What happens with the integrand at this value of x ?
15. Is the connection between the location of the minimum value of $\int_0^x \frac{t^2 - 13t + 22}{9} dt$ and the sign change of the integrand from negative to positive one you that you have seen before? If so, in what context?

CHAPTER 6**SE Applications of
Integration - TI****Chapter Outline**

- 6.1 THE AREA BETWEEN CURVES**
 - 6.2 VOLUME BY CROSS-SECTIONS**
 - 6.3 GATEWAY ARC LENGTH**
-

The activities below are intended to supplement our CK-12 Calculus FlexBook® textbook, [Chapter 5](#) .

6.1 The Area Between Curves

This activity is intended to supplement Calculus, Chapter 5, Lesson 1.

In this activity, you will explore:

- Using integrals to find the area between two curves.

Use this document to record your answers.

Problem 1 –Making Sidewalks

While integrals can be used to find the area under a curve, they can also be used to find the area between curves through subtraction (just make sure the subtraction order is the top curve minus bottom curve.)

Suppose you are a building contractor and need to know how much concrete to order to create a pathway that is $\frac{1}{3}$ foot deep. To find the volume, all that is needed is to multiply the area of the sidewalk by the depth of the sidewalk. The sidewalk borders can be modeled from -2π to 2π by:

$$f(x) = \sin(0.5x) + 3 \text{ and } g(x) = \sin(0.5x).$$

Graph both functions. Adjust the window settings to $-6.4 \leq x \leq 6.4$ and $-5 \leq y \leq 5$.

Now use the Integral tool from the Math menu to calculate the integrals of $f(x)$ and $g(x)$.

Enter -2π for the lower limit and 2π for the upper limit.

- What is the value of the integral of $f(x)$? Of $g(x)$?

On the Home screen, define $f(x)$ and $g(x)$. Then use the Numerical Integral command (**nInt**) in the Calc menu to find the area of the pathway.

Hint: the area is equal to the integral of $f(x) - g(x)$.

Note: The **nInt** command has the syntax: **nInt**(function, variable, left limit, right limit)

- What is the formula for the volume of the sidewalk?
- Now calculate how much concrete is needed for the pathway.

Problem 2 –Finding New Pathways

The owners have changed the design of the pathway. It will now be modeled from -2 to 2 by:

$$f(x) = x(x + 2.5)(x - 1.5) + 3$$

$$g(x) = x(x + 2)(x - 2)$$

Graph both functions. Adjust the window settings to $-4 \leq x \leq 4$ and $-5 \leq y \leq 9$.

Calculate the integrals of $f(x)$ and $g(x)$. Enter -2 for the lower limit and 2 for the upper limit.

- What is the value of the integral of $f(x)$? Of $g(x)$?
- Now calculate how much concrete is needed for the pathway on the Home screen. Remember to define $f(x)$ and $g(x)$.

Problem 3 –Stepping Stones

The owners also want stepping stones, which can be modeled by

$$f(x) = -(x-1)(x-2) + 2$$

$$g(x) = (x-1)(x-2) + 0.5.$$

This situation different because the starting and stopping points are not given. Assume that the stepping stones are $\frac{1}{3}$ foot thick.

Graph both functions. Adjust the window settings to $-1 \leq x \leq 7$ and $-4 \leq y \leq 4$ with a step size of 0.5 for both. Use the **Intersection** tool in the Math menu to find the intersection points. You can also use the **Solve** command on the Home screen.

- What are the coordinates for the two intersection points?

Calculate the integrals of $f(x)$ and $g(x)$. Use the x -values of the intersection points as the lower and upper limits.

- What is the value of the integral of $f(x)$? $g(x)$?
- Now calculate how much concrete is needed for the pathway on the Home screen. Remember to define $f(x)$ and $g(x)$.

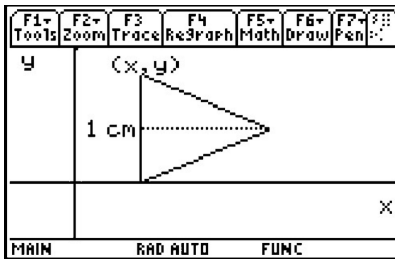
6.2 Volume by Cross-Sections

This activity is intended to supplement Calculus, Chapter 5, Lesson 2.

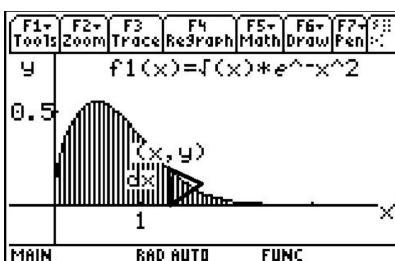
Part 1 –Setting Up The Problem And Understanding The Concept

A first step to solve calculus volume problems is to label the point and differential. You know the volume of an object is the area of the base times its height. So the differential dV equals area $\cdot dx$ or area $\cdot dy$.

- Typically the cross section is perpendicular to an axis. If the shape formed is perpendicular to the x -axis, what is the differential?
- The function may define the base with cross sections that form a variety of shapes.
 - What is the area of a square?
 - What is the area of a semicircle?
- Consider a function that defines the base of a solid where the cross sections perpendicular to the x -axis form equilateral triangles. Let the base of the triangle be parallel to the y -axis. What is the area of the triangle? Draw a sketch and justify your answer.
- If the length of one of the sides of this equilateral triangle is 1 cm, calculate the area. Show your calculation.



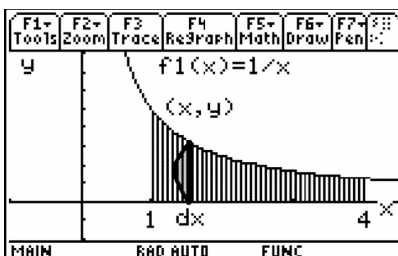
- Let the first quadrant region enclosed by the graph of $y = \sqrt{x} \cdot e^{-x^2}$ and the line $x = 2$ be the base of a solid. If the cross sections perpendicular to the x -axis are equilateral triangles, what is the volume of the solid? Show your work.



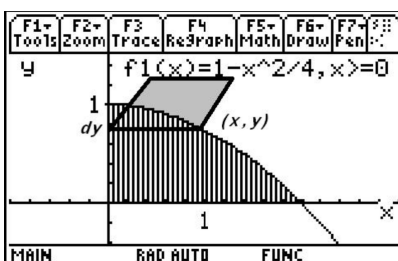
Part 2 –Homework

Questions 1 and 2 are non-calculator, exam-like problems. Show all your work. On Question 3, just show the set up and then use your calculator to find your answer.

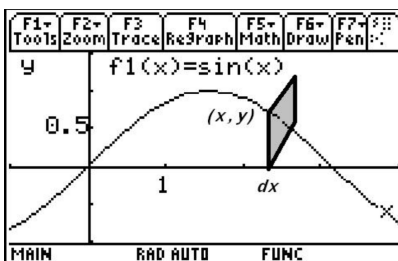
1. Let the first quadrant region enclosed by the graph of $y = \frac{1}{x}$ and the line $x = 1$ and $x = 4$ be the base of a solid. If the cross sections perpendicular to the x axis are semicircles, what is the volume of the solid? Show your work.



2. Let the base of a solid be the first quadrant region enclosed by the x -axis, the y -axis and the graph of $y = 1 - \frac{x^2}{4}$. If all the cross sections perpendicular to the y -axis are squares, what is the volume of the solid? Show your work.



3. Let the base of a solid be the first quadrant region enclosed by the x -axis and one arch of the graph $y = \sin(x)$. If all cross sections perpendicular to the x -axis are squares, then approximately what is the volume of the solid? Show your set up.



6.3 Gateway Arc Length

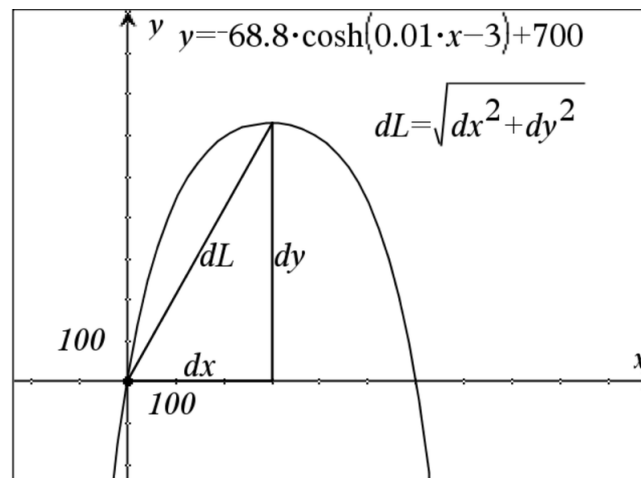
This activity is intended to supplement Calculus, Chapter 5, Lesson 3.

Part 1 –Arc Length Introduced

The Gateway to the West is an inverted catenary arch in St. Louis that is about 630 feet tall and 630 feet wide at its base. A catenary (a hyperbolic cosine function) is the shape that a chain or cable forms when it hangs between two points. The shape of the Gateway Arch can be modeled by the following equation:

$$y(x) = -68.8 \cosh(0.01x - 3) + 700.$$

1. If you were to ride in the elevator tram of the Gateway Arch, you would travel at least how far to get to the top? Explain.



Using the Pythagorean Theorem, we get $dL = \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2} dx$. As dx becomes smaller, so do dy and dL . As dL becomes smaller, the difference in length of dL and the length of the arc from x to $x + dx$ is eventually infinitesimal. So we can integrate both sides to give us the formula for arc length:

$$L = \int_a^b \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \right) dx.$$

2. On the Home screen, use the formula to find the arc length from $x = 0$ to $x = 300$ for $y(x) = -68.8 \cosh(0.01x - 3) + 700$. Write the formula and answer. Is this reasonable (when compared to your answer from Exercise 1)?

For parametric equations, the Pythagorean Theorem would yield $dL = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$. Integrating both sides gives us the arc length formula $L = \int_a^b \left(\sqrt{(x'(t))^2 + (y'(t))^2} \right) dt$.

Graph the parametric equations $x(t) = 2\cos(t)$ and $y(t) = 2\sin(t)$.

3. For the parametric equation $x(t) = 2\cos(t)$ and $y(t) = 2\sin(t)$, use the arc length formula to find the length from $t = 0$ to $t = \frac{\pi}{2}$. Show each step.

Now graph the equation $y_1(x) = \sqrt{4-x^2}$. When $x = 0$ to $x = 2$, this graph should look the same as the previous parametric curve.

4. Use the Home screen to find the arc length of $y_1(x) = \sqrt{4-x^2}$ from $x = 0$ to $x = 2$. Write out the equation and answer. Does this agree with the previous answer? Why or why not?

5. Graph $y_2(x) = x^2 - 9$ and approximate the arc length from $x = 0$ to $x = 3$. Write the arc length formula and solution for this arc length. Try using **arcLen**($y_2(x), x, 0, 3$) on the Home screen to check your answer.

6. Use the Pythagorean Theorem to approximate the arc length from $x = 0$ to $x = 3$ of $y = -x^2 + \frac{5}{3}x + 4$. On the Home screen, find the arc length using the formula. Write the formula and solution. Discuss if this is reasonable.

Part 2 –Additional Practice

1. Which of the following integrals gives the length of the graph of $y = \arcsin \frac{x}{2}$ between $x = a$ and $x = b$, where $0 < a < b < \frac{\pi}{2}$?

a. $\int_a^b \sqrt{\frac{x^2+8}{x^2+4}} dx$

b. $\int_a^b \sqrt{\frac{x^2+6}{x^2+4}} dx$

c. $\int_a^b \sqrt{\frac{x^2-2}{x^2-4}} dx$

d. $\int_a^b \sqrt{\frac{x^2-5}{x^2-4}} dx$

e. $\int_a^b \sqrt{\frac{2x^2+3}{x^2+1}} dx$

2. The length of the curve determined by the parametric equations $x = \sin t$ and $y = t$ from $t = 0$ to $t = \pi$ is

a. $\int_0^\pi \sqrt{\cos^2 t + 1} dt$

b. $\int_0^\pi \sqrt{\sin^2 t + 1} dt$

c. $\int_0^\pi \sqrt{\cos t + 1} dt$

d. $\int_0^\pi \sqrt{\sin t + 1} dt$

e. $\int_0^\pi \sqrt{1 - \cos t} dt$

3. Which of the following integrals gives the length of the graph of $y = \tan x$ between $x = a$ and $x = b$, where $0 < a < b < \frac{\pi}{2}$?

a. $\int_a^b \sqrt{x^2 + \tan^2 x} dx$

b. $\int_a^b \sqrt{x + \tan x} dx$

c. $\int_a^b \sqrt{1 + \sec^2 x} dx$

d. $\int_a^b \sqrt{1 + \tan^2 x} dx$

e. $\int_a^b \sqrt{1 + \sec^4 x} dx$

CHAPTER **7**

SE Transcendental Functions - TI

Chapter Outline

7.1 INVERSES OF FUNCTIONS

7.2 THE LOGARITHMIC DERIVATIVE

The activities below are intended to supplement our CK-12 Calculus FlexBook® textbook, [Chapter 6](#) .

7.1 Inverses of Functions

This activity is intended to supplement Calculus, Chapter 6, Lesson 1.

Exploring the Data

What happens when the domain and range of a function trade places? This special “trade” creates inverses of functions, which will be explored in this activity.

To the right is wind tunnel data obtained from www.doe.state.la.us.

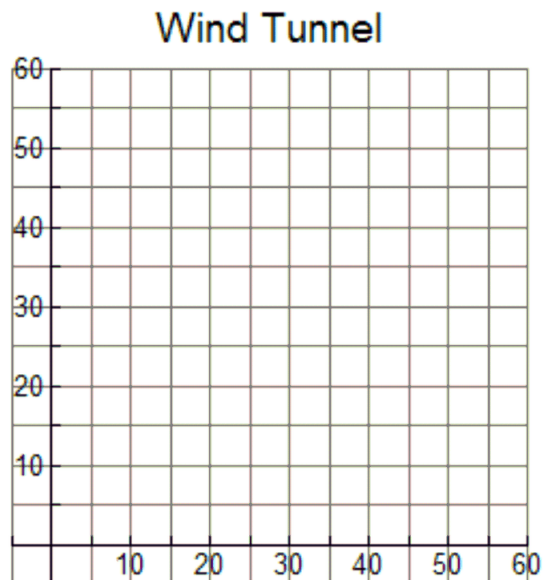
Wind tunnel experiments are used to test the wind friction or resistance of an automobile at different speeds. The given data shows wind speed in miles per hour versus resistance in pounds.

Record this data in $L1$ and $L2$ by pressing **STAT** and select **Edit**. Make sure to clear the lists before you input any data.

TABLE 7.1:

Speed (MPH)	Resistance (lbs.)
10	6.4
21	9.2
34	17.0
40	22.4
45	30.2
55	59.2

1. Construct a scatter plot and graph your data on the grid below. Recall that to construct a scatter plot, press 2^{nd} $Y=$, select the scatter plot option, and make sure that $L1$ and $L2$ are selected for Xlist and Ylist. Press **ZOOM** and select the **ZoomStat** option.
2. Now graph a second scatter plot on your calculator (Plot 2) and graph the new data on the grid above. For this graph, you will switch the domain and range. This means that in the StatPlot menu, $L1$ and $L2$ trade places.



Exploring the Problem

- After graphing the original wind tunnel data and then graphing the data with the domain and range switched, what did you notice about the graphs of the two sets of points?
- Find the midpoint between the first points from the first and second scatter plots and the midpoint of the last points from the first and second scatter plots.
- Find the equation of the line that connects these two midpoints. Graph this line on the grid above and on your calculator. What does this line represent?

Developing the Pattern Further

Clear all data in $L1$ and $L2$. Also, clear all plots; Plot1, Plot2, and $Y1$.

- Graph $y = 2x + 3$ in a standard viewing window. Use the **zero** command (2^{nd} **TRACE**) to find the x -intercept. Use **TRACE** to find the y -intercept.
- Switch the locations of the x - and y -coordinates of these points to (y, x) and input these as $L1$ and $L2$. Graph these two points using **Plot1**. Find the equation of the line connecting these two points and graph it in $Y2$.
- Switch x and y in the equation $y = 2x + 3$ and solve for y . How does your result compare to your answer to Question 7?
- List 3 ways in which the inverse of a function may be obtained.
- The inverse of a function is always a function. Agree Disagree

Explain your answer choice and provide at least one example to illustrate why you chose it.

- Find the inverse of the following functions if an inverse exists. If an inverse does not exist, write “does not exist.”

TABLE 7.2:

Function	Inverse
$f(x) = 6x - 2$	$f^{-1}(x) =$
$f(x) = \frac{1}{2}x - \frac{3}{4}$	$f^{-1}(x) =$
$ f(x) = x $	$f^{-1}(x) =$

7.2 The Logarithmic Derivative

This activity is intended to supplement Calculus, Chapter 6, Lesson 3.

Problem 1 –The Derivative of

If (x, y) is a point on $y = f(x)$ and $y = g(x)$ is the inverse of $f(x)$, then (y, x) is a point on $g(x)$. We know that $e^0 = 1$ and $\ln(1) = 0$, so $(0, 1)$ is a point on $y = e^x$ and $(1, 0)$ is a point on $y = \ln(x)$. We could do this for several points and keep getting the same inverse results.

Thus, if $y = e^x$, then $x = e^y$ will be equivalent to $y = \ln(x)$ because they are inverses of one another. Now we can take the implicit derivative with respect to x of $x = e^y$.

$$x = e^y \rightarrow 1 = \frac{dy}{dx} \cdot e^y \rightarrow 1 = \frac{dy}{dx} \cdot x \rightarrow \frac{1}{x} = \frac{dy}{dx}$$

Use the **limit** command to test this formula. Be careful with your parentheses.

- Find $\lim_{h \rightarrow 0} \frac{\ln(2+h) - \ln(2)}{h}$.
- Do the same with $\lim_{h \rightarrow 0} \frac{\ln(3+h) - \ln(3)}{h}$.
- What is $\lim_{h \rightarrow 0} \frac{\ln(x+h) - \ln(x)}{h}$?

Use the **derivative** command to find the derivative of the logarithmic function $f(x) = \ln(x)$.

Problem 2 –The Derivative of

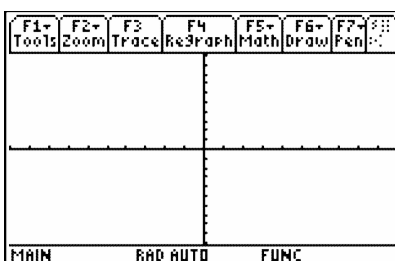
What happens if our logarithm has a base other than e ? We need to know how to take the derivative of the function $y = \log_a(x)$.

First we want to compare $y_1 = \ln(x)$ and $y_2 = \log_2(x)$.

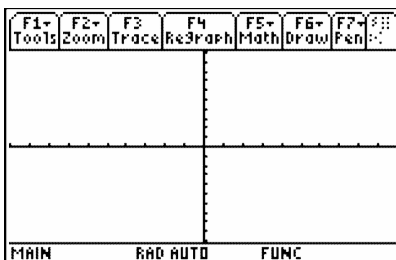
To enter $\log_2(x)$, use the alpha keys to spell out **log**.

Within the parentheses, enter the expression, then the base.

- Graph both functions ($\ln(x)$ and $\log_2(x)$) on the same set of axes. Sketch your graph to the right. What do you notice?



- Do the same steps with $y1 = \ln(x)$ and $y3 = \log_4(x)$. What do you notice?



- Simplify the following ratios.

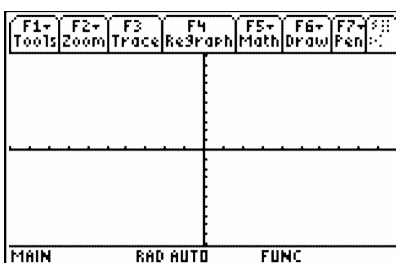
$$\frac{\ln(x)}{\log_2(x)}$$

$$\frac{\ln(x)}{\log_4(x)}$$

$$\frac{\ln(x)}{\log_a(x)}$$

Sometimes the ratio $\frac{\ln(x)}{\log_a(x)}$ is written as $\ln(x) = \ln(a) \cdot \log_a(x)$. We can rewrite this ratio as $\log_a(x) = \frac{\ln(x)}{\ln(a)}$ and call it an identity.

- Graph the following functions on the same set of axes: $y1 = \ln(x)$, $y2 = \ln(2) \cdot \log_2(x)$, $y3 = \ln(3) \cdot \log_3(x)$. What was the result?



What happens when we take the derivative of $y = \log_a(x)$. Use the **derivative** command to find the derivatives of the functions below.

$$f(x) = \log_2(x)$$

$$g(x) = \log_3(x)$$

$$h(x) = \log_a(x)$$

- Do you notice a pattern?

What does $\log_2(e)$ equal? If we use the formula from earlier in this class, we get $\log_2(e) = \frac{\ln(e)}{\ln(2)} = \frac{1}{\ln(2)}$.

Therefore, the general result is $y = \log_a(x) \rightarrow \frac{dy}{dx} = \frac{1}{(x \ln(a))}$.

Problem 3 –Derivative of Exponential and Logarithmic Functions Using the Chain Rule

Now we want to take the derivative of more complicated functions:

Recall: $y = a^u \rightarrow \frac{dy}{dx} = a^u \frac{du}{dx}$ where u depends on x .

- Suppose that $y = \log_a(u)$, where u depends on x . Using the chain rule, take the derivative of this function.

Find the derivative of the following functions with the chain rule.

Identify $u(x)$ and a for each function before you find the derivative.

- $f(x) = 5^{(x^2)}$ $u(x) = \underline{\hspace{2cm}}$ $a = \underline{\hspace{2cm}}$

$$f'(x) = \underline{\hspace{2cm}}$$

- $g(x) = e^{(x^3+2)}$ $u(x) = \underline{\hspace{2cm}}$ $a = \underline{\hspace{2cm}}$

$$g'(x) = \underline{\hspace{2cm}}$$

- $h(x) = \log_3(x^4 + 7)$ $u(x) = \underline{\hspace{2cm}}$ $a = \underline{\hspace{2cm}}$

$$h'(x) = \underline{\hspace{2cm}}$$

- $j(x) = \ln\left(\sqrt{x^6+2}\right)$ $u(x) = \underline{\hspace{2cm}}$ $a = \underline{\hspace{2cm}}$

$$j'(x) = \underline{\hspace{2cm}}$$

CHAPTER 8 SE Integration Techniques - TI

Chapter Outline

- 8.1 INTEGRATION BY SUBSTITUTION
 - 8.2 INTEGRATION BY PARTS
 - 8.3 CHARGED UP
-

The activities below are intended to supplement our CK-12 Calculus FlexBook® textbook, [Chapter 7](#) .

8.1 Integration by Substitution

This activity is intended to supplement Calculus, Chapter 7, Lesson 1.

In this activity, you will explore:

- *Integration of standard forms*
- *Substitution methods of integration*

Use this document to record your answers. Check your answers with the **Integrate** command.

Problem 1 –Introduction

1. Consider the integral $\int \sqrt{2x+3} dx$. Let $u = 2x+3$. Evaluate the integral using substitution.

Use the table below to guide you.

TABLE 8.1:

$f(x) =$	$\sqrt{2x+3}$
$u =$	$2x+3$
$du =$	
$g(u) =$	
$\int g(u) du =$	
$\int f(x) dx =$	

2. Try using substitution to integrate $\int \sin(x) \cos(x) dx$. Let $u = \sin(x)$.

3. Now integrate the same integral, but let $u = \cos(x)$. How does the result compare to the one above?

4. $\sin(x) \cos(x) dx$ can be rewritten as $\frac{1}{2} \sin(2x)$ using the Double Angle formula.

What is the result when you integrate $\int \frac{1}{2} \sin(2x)$ using substitution?

Problem 2 –Common Feature

Find the result of the following integrals using substitution.

5. $\int \frac{x+1}{x^2+2x+3} dx$

6. $\int \sin(x) e^{\cos(x)} dx$

7. $\int \frac{x}{4x^2+1} dx$

8. What do these integrals have in common that makes them suitable for the substitution method?

Extension

Use trigonometric identities to rearrange the following integrals and then use the substitution method to integrate.

9. $\int \tan(x) dx$

10. $\int \cos^3(x)$

8.2 Integration by Parts

This activity is intended to supplement Calculus, Chapter 7, Lesson 2.

In this activity, you will explore:

- *product rule of differentiation*
- *integration by parts*

Use this document to record your answers.

Exercises

1. State the product rule for a function of the form $u(x) \cdot v(x)$.
2. Apply the product rule to the function $\sin(x) \cdot \ln(x)$.
3. Do you agree or disagree with the following statement? Explain.

$$\int \frac{d}{dx}(f(x))dx = \frac{d}{dx} \left(\int f(x)dx \right) = f(x)$$

4. What is the integral of the left side of the product rule?

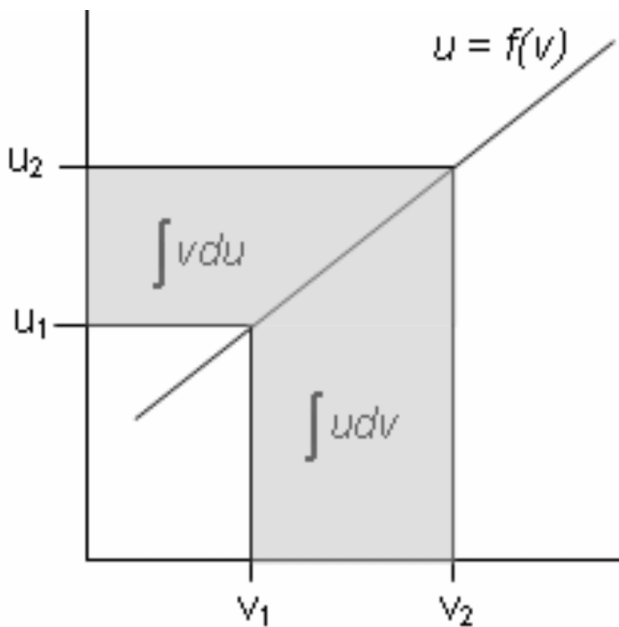
$$\int \left(\frac{d}{dx}(u(x) \cdot v(x)) \right) dx =$$

5. What is the integral of the right side?

$$\int \left(u(x) \cdot \frac{dv}{dx} + v(x) \cdot \frac{du}{dx} \right) dx =$$

6. Explain the relationship between the areas shown on the graph and the following equation:

$$\int_{v_1}^{v_2} u \cdot dv = u \cdot v - \int_{u_1}^{u_2} v \cdot du$$



7. Use the method of **integration by parts** to compute the integral of $\ln(x)$.

Remember the formula for Integration by parts is $\int u \cdot dv = u \cdot v - \int v \cdot du$

$$\int \ln(x) \cdot 1 \, dx \rightarrow u = \ln(x) \text{ and } dv = 1 \, dx$$

$$du = \quad \quad \quad v =$$

Result =

Check by integration directly. (**Home >F3:Calc >2:Integrate**) or (**Home > 2nd 7**)

Consider the function $f(x) = \sin(\ln(x))$.

$$u = \sin(\ln(x)) \rightarrow du = \frac{\cos(\ln(x))}{x} dx$$

$$dv = dx \rightarrow v = x(+C)$$

$$\begin{aligned} \int \sin(\ln(x)) \cdot 1 \, dx &= x \cdot \sin(\ln(x)) - \int x \cdot \frac{\cos(\ln(x))}{x} dx(+C) \\ &= x \cdot \sin(\ln(x)) - \int \cos(\ln(x)) dx(+C) \end{aligned}$$

8. Find $\int \cos(\ln(x)) dx$.

$$u = \quad \quad \quad du = \quad \quad \quad dv = \quad \quad \quad v =$$

$$\int \cos(\ln(x)) dx =$$

9. Substitute the result for $\cos(\ln(x))$ into the result for $\sin(\ln(x))$.

$$u = \quad du = \quad dv = \quad v =$$
$$\int \sin(\ln(x)) dx =$$

10. Use integration by parts to solve the following. If you need to use integration by parts more than once, do so. Check your result.

a. $\int \tan^{-1}(x) dx$

b. $\int x^2 \cdot e^x dx$

c. $\int x \cdot \tan^{-1}(x) dx$

d. $\int x \cdot \cos(2x + 1) dx$

11. **(Extension 1)** Does it matter in which order $u(x)$ and $v(x)$ are selected for the method of integration by parts?

12. **(Extension 2)** Is there likely to be an integration rule based upon the quotient rule just as Integration by Parts was based upon the product rule?

8.3 Charged Up

This activity is intended to supplement Calculus, Chapter 7, Lesson 7.

Part 1 –Separable Differential Equations Introduced

1. A capacitor, like one used for a camera flash, is charged up. When it discharges rapidly the rate of change of charge, q , with respect to time, t , is directly proportional to the charge. Write this as a differential equation.

The first step is to separate the variables, and then integrate and solve for y .

2. Find $y(0)$, if $\frac{dy}{dx} = \sin(x) \cos^2(x)$ and $y\left(\frac{\pi}{2}\right) = 0$. After integrating, use the initial condition $y = 0$ when $x = \frac{\pi}{2}$ to find the constant of integration. Then, substitute $x = 0$ to find $y(0)$.

Let's return to the capacitor. Now that it is discharged, we need to get it charged up again. A 9V battery is connected to a $100k\Omega$ resistor, R , and $100\mu F$ capacitor, C .

The conservation of energy gives us the differential equation $\frac{dq}{dt} \cdot R = V - \frac{q}{C} \rightarrow \frac{dq}{dt} \cdot R \cdot C = V \cdot C - q$.

After substituting the given information and simplifying, we get the differential equation $10\frac{dq}{dt} = 0.9 - q$.

3. For the differential equation $10\frac{dq}{dt} = 0.9 - q$, separate the variables and integrate.

4. Apply the initial condition when time = 0, the charge $q = 0$ and solve for q .

The syntax for **deSolve** is **deSolve**($y' = f(x,y), x, y$) where x is the independent and y is the dependent variable. The **deSolve** command can be found in the **HOME** screen by pressing \square and selecting **C:deSolve**(.

5. On the **HOME** screen, type **deSolve**($10q' = 0.9 - q$ and $q(0) = 0, t, q$). Write down this answer and reconcile it with your previous solution.

6. In the **HOME** screen enter **deSolve**($y' = y/x, x, y$) to find the general solution of $\frac{dy}{dx} = \frac{y}{x}$. Write the answer. Show your work to solve this differential equation by hand and apply the initial condition $y(1) = 1$ to find the particular solution.

Part 2 –Homework/Extension –Practice with deSolve and Exploring DEs

Find the general solution for the following separable differential equations. Write the solution in an acceptable format, (for example, use C instead of @ 7). Show all the steps by hand if your teacher instructs you to do so.

1. $y' = k \cdot y$

2. $y' = \frac{x}{y}$

3. $y' = \frac{2x}{y^2}$

4. $y' = \frac{3x^2}{y}$

Open the **GDB** graph labeled *diffq1*. Observe the family of solutions to the differential equation from Question 4, $y' = \frac{3x^2}{y}$. Many particular solutions can come from a general solution. When you are finished viewing the family of functions, go to the $Y =$ screen and delete the function.

5. Not all differential equations are separable. Use **deSolve** to find the solution to the non-separable differential equation $x \cdot y' = 3x^2 + 2 - y$. What does this graph look like if the integration constant is 0? Explain. Open picture *diffq2* to view graph.

Find the particular solution for the following equations. Show your work. Solve for y . Explore other DEs on your own. Do you get any surprising results?

6. $y' = x \cdot y^2$ and $y(0) = 1$

7. $y' = 1 + y^2$ and $y(0) = 1$

8. $y' = 7y$ and $y(0) = \ln(e)$

CHAPTER

9

SE Infinite Series - TI**Chapter Outline**

9.1 EXPLORING GEOMETRIC SEQUENCES

9.2 INFINITE GEOMETRIC SERIES

9.3 MR. TAYLOR, I PRESUME?

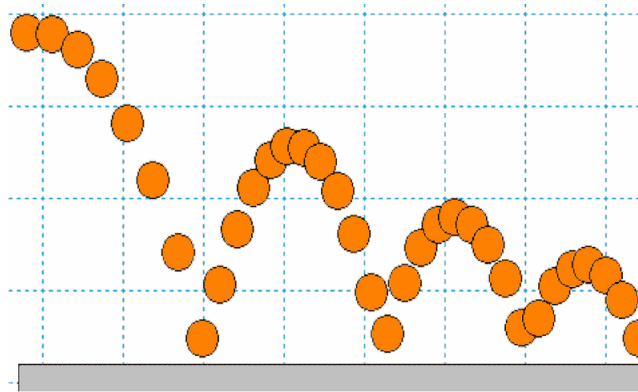
9.4 AP CALCULUS EXAM PREP

The activities below are intended to supplement our CK-12 Calculus FlexBook® textbook, [Chapter 8](#) .

9.1 Exploring Geometric Sequences

This activity is intended to supplement Calculus, Chapter 8, Lesson 1.

The height that a ball rebounds to after repeated bounces is an example of a geometric sequence. The top of the ball appears to be about 4.0, 2.8, 2.0, and 1.4 units. If the ratios of consecutive terms of a sequence are the same then it is a geometric sequence. The common ratio r for these values is about 0.7.



Changing the Common Ratio

Explore what happens when the common ratio changes.

Start the **Transform App**. Press $Y =$ and for Y_1 , enter $4 * A^{(X - 1)}$.

Change your settings by pressing **WINDOW** and arrow right to go to **SETTINGS**. Set $A = 0.7$ and **Step** = 0.1.

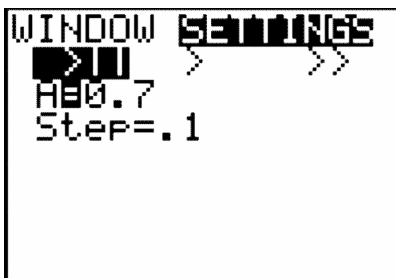
Graph the function by pressing **ZOOM** and selecting ZoomStandard. Change the value of the common ratio (A).

1. What did you observe happens when you change the common ratio from positive to negative? Explain why this happens.

```

Plot1 Plot2 Plot3
M Y1 = 4 * A ^ (X - 1)
M Y2 =
M Y3 =
M Y4 =
M Y5 =
M Y6 =
M Y7 =

```



- When the common ratio is larger than 1, explain what happens to the graph and values of y .
- What r -values could model the heights of a ball bounce? Explain.

Changing the Initial Value

Press $Y =$ and change $Y1$ to $B \cdot A^{(X-1)}$.

Change the **SETTINGS** so that $A = 0.7$, the $B = 4$ and Step = 0.1.

- Explain your observations of what happens when B changes. What is B also known as?
- If the common ratio is less than -1, describe what occurs to the terms of the sequence.

Extension –Partial Sum Formula

The sum of a finite geometric series can be useful for calculating funds in your bank account, the depreciation of a car, or the population growth of a city.

e.g. $S_6 = 4 + 8 + 16 + 32 + 64 + 128$

In this example the common ratio is 2, the first term is 4, and there are 6 terms.

The general formula

$$S_n = a_1 + a_2 + a_3 + \dots + a_{n-1} + a_n$$

Since $a_n = r \cdot a_{n-1}$, substituting gives

$$\begin{aligned} S_n &= a_1 + r \cdot a_1 + r^2 \cdot a_1 + r^3 \cdot a_1 + \dots + r^{n-2} \cdot a_1 + r^{n-1} \cdot a_1 \\ r \cdot S_n &= r \cdot a_1 + r^2 \cdot a_1 + r^3 \cdot a_1 + \dots + r^{n-1} \cdot a_1 + r^n \cdot a_1 \end{aligned}$$

Subtracting the previous two lines

$$S_n - r \cdot S_n = a_1 - r^n \cdot a_1$$

$$S_n(1 - r) = a_1(1 - r^n) \text{ So } S_n = a_1 \cdot \frac{1 - r^n}{1 - r}$$

Use the formula to find the sum of the following finite geometric series.

6. Find S_5 for $a_n = 6\left(\frac{1}{3}\right)^{n-1}$. =

7. $\frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \frac{1}{7^4} + \frac{1}{7^5} + \frac{1}{7^6} =$

8. Find S_{25} for $a_n = 2(1.01)^{n-1}$.

9. $64 - 32 + 16 - 8 + 4 - 2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} + \frac{1}{256} =$

9.2 Infinite Geometric Series

This activity is intended to supplement Calculus, Chapter 8, Lesson 2.

Problem 1 –Investigating Infinite Geometric Series

Explore what happens when the common ratio changes for an infinite geometric series.

For each of the r -values in the table, you need to

- create lists $L2$ and $L3$
- determine if the series converges
- if it does converge, create and view the scatter plot

Press **STAT** Edit... to open the list editor.

Create list $L1$ with values 1 to 20 by entering **seq**($X, X, 1, 20, 1$) in the top of the $L1$ field.

Create list $L2$ with values of $ar^{n-1} = 200r^{n-1}$ by entering $200 * r^{(L1 - 1)}$ with different values of r based on the table below. $200 * (-2)^{(L1 - 1)}, 200 * (-0.5)^{(L1 - 1)}$, etc.

Create list $L3$ with the partial sums by entering **cumSum**($L2$).

Note: Press 2nd **STAT**, arrow right to the **OPS** menu for the **seq**(and **cumSum**(commands.

- Look at the partial sums (in $L3$) and determine if the sums converge to a number or diverge to infinity (or negative infinity) as n gets large.

Record your results in the table. If it converges, then what does it appear to converge to?

TABLE 9.1:

r	-2	-0.5	-0.25	0.25	0.5	2
Converges or Diverges						

Press 2nd $Y =$, and select **Plot1**. Create a scatter plot, choosing $L1$ for x and $L3$ for y . Press **ZOOM** and select **ZoomData**.



- For what range of values of r does the infinite geometric series converge?
- What do you notice about the scatter plot when the series converges?

Problem 2 –Deriving a Formula for the Sum of a Convergent Infinite Geometric Series

Recall that a finite geometric series is of the form

$$S_n = a_1 + r \cdot a_1 + r^2 \cdot a_1 + r^3 \cdot a_1 + r^4 \cdot a_1 + \dots + r^n \cdot a_1 = \sum_{i=1}^n a_1 r^{i-1} = \frac{a_1(1-r^n)}{1-r}$$

4. Let $r = 0.7$. Use the Home screen to complete the following table.

TABLE 9.2:

n $r^n = 0.7^n$	10	100	1000	10000
----------------------	----	-----	------	-------

If $|r| < 1$, then what is the value of r^n as n gets very large?

5. How does the formula $S_n = \frac{a_1(1-r^n)}{1-r}$ change, if n goes to infinity (gets very large)?

Therefore, if $|r| < 1$, then the infinite geometric series of the form $S = a_1 + r \cdot a_1 + r^2 \cdot a_1 + r^3 \cdot a_1 + r^4 \cdot a_1 + \dots$ converges and has the sum $S = \frac{a_1}{1-r}$.

Problem 3 –Apply what was learned

Use the formulas for the sums of finite and infinite geometric series to complete this problem.

6. A patient is prescribed a 240 mg dose of a long-term, pain-reducing drug that should be taken every 4 hours. It is known that after each hour, 15% of the original dosage leaves the body. Under these conditions, the amount of drugs remaining in the body (at 4-hour intervals) forms a geometric series.

- What is the common ratio of the geometric series?
- How many milligrams of the drug are present in the body after 4 hours (2^{nd} dosage)?
- Complete the table for the amount of the drug in the body for several 4 hour intervals.

TABLE 9.3:

Hours	0 (1st dosage)	4 (2^{nd} dosage)	8 (3^{rd} dosage)	12	16
Amount in the Body					

- How many milligrams of the drug are in the body after 24 hours?
- How many milligrams of the drug are in the body after 72 hours?
- How many milligrams of the drug are in the body after t hours?
- The minimum lethal dosage of the pain-reducing drug is 600 mg. Will the patient ever have this much of the drug in his or her system if he or she continuously (infinitely) takes the drug every four hours?
- If the patient decides to take the drug every 2 hours, against the doctor's orders, then will the patient reach the

minimum lethal dosage?

9.3 Mr. Taylor, I Presume?

This activity is intended to supplement Calculus, Chapter 8, Lesson 7.

Introduction

Polynomials are great things to work with. They are easy to integrate, to differentiate, and even to tell jokes to (they always laugh!). Wouldn't it be nice if it were possible to transform a very difficult function into a nice, easygoing polynomial? Of course it would! But how?

Believe it or not, it is possible to determine any polynomial of the form $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n$ by just knowing the value of its derivatives at a point.

Taylor polynomials centered at zero

For example, find a polynomial $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ if $P(0) = 1, P'(0) = 1, P''(0) = 6$ and $P'''(0) = 9$.

The first step is to find the derivatives:

$$P'(x) = a_1 + 2a_2x + 3a_3x^2$$

$$P''(x) = 2a_2 + 6a_3x$$

$$P'''(x) = 6a_3$$

Since we know the value of each derivative when $x = 0$, we can determine the a_n terms

$$P'(0) = 1 = a_1 + 2a_2(0) + 3a_3(0)^2$$

$$P''(0) = 6 = 2a_2 + 6a_3$$

$$P'''(0) = 9 = 6a_3$$

$$1 = a_1$$

$$6 = 2a_2 \rightarrow \frac{6}{2} = a_2$$

$$9 = 6a_3 \rightarrow \frac{9}{6} = a_3$$

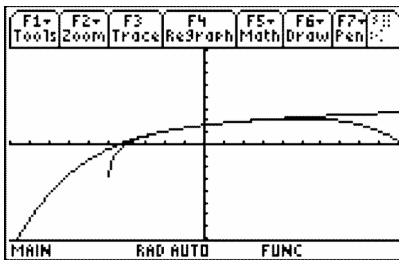
So the polynomial will be $P(x) = 1 + 1x + \frac{6}{2}x^2 + \frac{9}{6}x^3$.

Notice that the number in the numerator of the a_n term is $f^{(n)}(0)$ and the denominator is $n!$. This will come in handy in the next exercise.

While this is amazing, it is important to note that this polynomial is centered at the value $x = 0$ because that is where we calculated the derivative values.

Enough chatter, let's get to it.

Find a polynomial of degree four that approximates $f(x) = \ln(x + 5)$ when $x = 0$.



1. Find the values of $f(x), f'(x), f''(x), f'''(x), f^{(4)}(x)$ when $x = 0$:

$f(0) =$

$f'(0) =$

$f''(0) =$

$f'''(0) =$

$f^{(4)}(0) =$

2. Substitute the derivative values into the numerator and the $n!$ into the denominator of each term in $P(x)$. Simplify the polynomial.

$P_4(x) =$

Check your answer with the **Taylor** command. We use the value of the center for a (in this case 0).

Home > F3:Calculus > 9:Taylor > (exp, var, power, a)

3. Set up a table for $y1(x) = P_4(x)$ and $y2(x) = \ln(x + 5)$ where x starts with -4 and increases by 1 in the table.

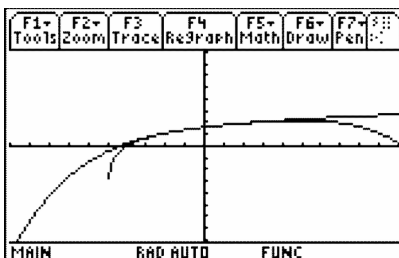
Decide where the polynomial and the function agree or nearly agree in value.

x	y1	y2
-2.	1.1017	1.0986
-1.	1.3864	1.3863
0.	1.6094	1.6094
1.	1.7917	1.7918
2.	1.9444	1.9459

x = -2.

Notice that the values of the Taylor polynomial and the values of the function do not agree everywhere on the graph. In fact, they are closest where the derivative was evaluated. This is called the *center* of the polynomial.

4. On what interval does the polynomial best approximate the original function?



5. Use the **Taylor** command and increase the power. Try several larger powers. What do you notice about the interval as the degree of the polynomial changes?

Taylor polynomial not centered at zero

Now it is time to leave the origin and find Taylor polynomials whose centers are not zero. The only adjustment is to change x^n to $(x - a)^n$ where a is the center of the approximation.

We want to find a 4th degree Taylor polynomial for $f(x) = \frac{1}{2-x}$ centered at $x = 1$.

6. Fill in the following chart.

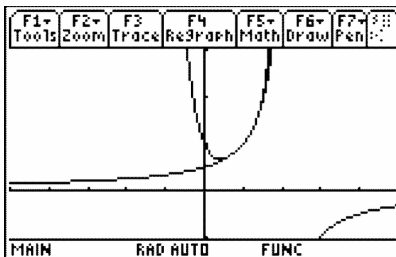
TABLE 9.4:

	Derivative	Value at $x = 1$
$f(x)$		
$f'(x)$		
$f''(x)$		
$f'''(x)$		
$f^{(4)}(x)$		

Now substitute the derivative values into the numerator and $n!$ into the denominator of each term in $P(x)$. Remember to write $(x - 1)^n$ instead of x^n . Simplify the polynomial.

$P(x) =$

Graph the function and the Taylor polynomial you just found. Examine the graphs and pay close attention to where $x = 1$.



7. Your teacher will give you three new values of a center. Find the 4th degree Taylor polynomial using the **Taylor** command with these values as the center. Graph both the Taylor polynomial and the original function for each value. What do you notice about the interval as the center is changed?

9.4 AP Calculus Exam Prep

This activity is designed for AP Calculus students as a review for the AB Calculus Exam at the end of the year.

Part 1 – Calculator Active Exam Tips

The AP* Calculus exam is long. It is a marathon of the mind. Success requires training.

Tip #1 – Know the course

There is a free-response (FR) section and a multiple-choice (MC) section. For both there is a 'calculator NOT allowed' section and a 'graphing calculator is required' section.

On the two FR sections you have a total 1 hour and 30 minutes to do 6 questions, 3 with your TI-89 handheld and 3 without. In Section 1 Part A, you will have 55 minutes to complete 28 MC questions where a calculator is NOT allowed. Section 1 Part B requires a calculator to complete 17 MC questions in 50 minutes.

1. About how much time should you spend on each question in each part of the exam?

Tip #2 – Know the expectations

On the calculator active questions you should:

- Give answers to 3 decimal places (unless it says otherwise, e.g., give answer to the nearest cent).
- Be able to do the following four things:
 - graph on an arbitrary window
 - solve equations (find zeros and intersection points)
 - numerically calculate the derivative
 - numerically calculate the definite integral

2. The position function of a particle is $s(t) = 3t + 6.5 \sin(2t)$. What is $v(2)$?

On the **HOME** screen, enter $d(3t + 6.5 \sin(2t), t) | t = 2$. You can access the derivative command by selecting **F3:Calc >1: d (differentiate)**, or by using the shortcut $2^{nd} [d]$. The “such that” bar after the derivative command is input by pressing the $[|]$ key.

Notice the mode settings at the bottom of your screen. It should say RAD indicating that you are in radian mode. To change the angle mode, go to **MODE >Angle**. You will only need to be in radian mode for the entire AP exam.

Using **AUTO** mode under **MODE >Exact/Approx** is also nice. You get an exact answer if possible, or if there is a decimal in what you enter, you get a decimal answer. Pressing $[\blacklozenge]$ **[ENTER]** (\approx) will give an approximation.

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Tip #3 – Have the right gear

Besides the right settings, bring a couple of pencils. You can actually bring 2 graphing calculators. Be sure they have reasonably fresh batteries.

3. $h(x) = \int_0^x (t \cos(t^3) - 0.7) dt$. Where is $h(x)$ increasing on the interval $-1.8 < x < 1.4$?

Find the solution graphically. Press $[\blacklozenge]$ **[Y=]** and enter the function in $y1(x)$. The integration command can be

entered by pressing $2 + <$. Change the window to view only the domain by pressing $[2^{nd}] [f]$ and entering **XMin:** -1.8 and **XMax:** 1.4. Graph the derivative. Why? What are you looking for?

Find the zeros by selecting **F5:Math >2:Zero**. You can trace to input your lower and upper limits to find each root or enter your lower and upper limits on the keypad.

Try solving for the zeros algebraically. Press **[HOME]** to return to the **HOME** screen. Then press **F2:Algebra >Solve**. If you have already entered the function in $y1(x)$ and the derivative of the function in $y2(x)$, enter $\text{solve}(y2(x) = 0, x) | -1.8 < x < 1.4$.

The 'such that' symbol needs to be outside of the parentheses.

4. Find the area of the region bounded by $y = \frac{1}{x}$, $y = 3 \ln(x)$, and the vertical line $x = 3$.

One way to find the intersection point is to graph the functions using a suitable window and then selecting **F5:Math >5:Intersection**. Use the **ENTER** key and the arrow keys to select the two curves and input a lower and upper bound. To store a value, like the x -coordinate of the point of intersection, press the **[STO→]** key. This will store the x -coordinate of the point of intersection as xc . Then, use xc in the limits of integration. By doing this, you won't make a rounding error and you won't make a mistake in typing the numbers. Next, return to the **HOME** screen and solve the integral.

Part 2 – AB Exam Practice Non-Calculator Type Questions

Answer the following AP-type exam questions. These are similar to non-calculator questions you will encounter on the exam. You should spend on average less than 2 minutes for each question. Don't spend too much time on any one question. On the actual exam, you will have plenty of space so extra paper will not be needed. However, for this practice you may need additional space.

1. If $y = (4x^2 + 5)^3$, then $\frac{dy}{dx} =$

- (A) $12(8x)^2$
- (B) $4x(4x^2 + 5)^3$
- (C) $8x(4x^2 + 5)^2$
- (D) $24x(4x^2 + 5)^2$
- (E) $12x(4x^2 + 5)^2$

2. $\int_0^3 \left(\frac{1}{3}e^{-\frac{x}{3}}\right) dx$

- (A) $\frac{1}{9}(1 - e^{-1})$
- (B) $e^{-1} - 1$
- (C) e^{-1}
- (D) $\frac{1}{3}(1 - e^{-1})$
- (E) $1 - e^{-1}$

3. If $\tan(x) = e^y$, what is $\frac{dy}{dx}$?

- (A) $\frac{\sec(x)}{\tan(x)}$
- (B) $\frac{1}{\sin(x)\cos(x)}$
- (C) $\frac{\cos(x)}{\sin(x)}$

(D) $\ln(\tan(x))$

(E) $\tan(x) \sec^2(x)$

4. If $y = \frac{5x+7}{2x+3}$, then $\frac{dy}{dx} =$

(A) $\frac{-10x-14}{(2x+3)^2}$

(B) $\frac{29}{(2x+3)^2}$

(C) $\frac{1}{(2x+3)^2}$

(D) $\frac{22}{(2x+3)^2}$

(E) $\frac{5}{2}$

5. $\int_0^{\frac{\pi}{6}} (2 \cos x) dx$

(A) $\sqrt{3}$

(B) $\sqrt{3} - 2$

(C) -1.5

(D) 1

(E) $\frac{\sqrt{3}}{2}$

6. $\lim_{x \rightarrow \infty} \left(\frac{4x^2 - 5x - 3}{3x^2 + 2x + 4} \right)$

(A) $-\frac{4}{9}$

(B) $-\frac{3}{4}$

(C) 1

(D) $\frac{4}{3}$

(E) ∞

7. If $f(x) = 1 + 3g(x)$ when $2 \leq x \leq 7$, find $\int_2^7 (f(x) - g(x)) dx$.

(A) $4 \int_2^7 (g(x)) dx$

(B) $x + 2 \int_2^7 (g(x)) dx$

(C) $5 + 2 \int_2^7 (g(x)) dx$

(D) $-\int_2^7 (1 + 3g(x)) dx$

(E) $-10 - 2 \int_2^7 (g(x)) dx$

8. If $y = \sqrt{2x} \cdot \tan(3x)$, then $\frac{dy}{dx} =$

(A) $\frac{1}{2}(2x)^{-\frac{1}{2}} \sec^2(3x)$

(B) $3(2x)^{-\frac{1}{2}} \sec^2(3x)$

(C) $\sqrt{2x} \sec^2(3x) + \frac{1}{2}(2x)^{-\frac{1}{2}} \tan(3x)$

(D) $3\sqrt{2x} \sec^2(3x) + (2x)^{-\frac{1}{2}} \tan(3x)$

(E) $\sqrt{2x} \sec^2(3x) + (2x)^{-\frac{3}{2}} \tan(3x)$

9. Find the equation of the tangent line to $y = \sin(2x)$ at the point $(\pi, 0)$.

(A) $y = 2x - 2\pi$

(B) $y = -2x + 2\pi$

(C) $y = 2x$

(D) $y = x - \pi$

(E) $y = x - 2\pi$